Catastrophe Aversion and Risk Equity under Dependent Risks

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Abstract

Catastrophe aversion and risk equity are important concepts in both risk management theory and practice. Keeney (1980) was the first to formally define these concepts. He demonstrated that the two concepts are always in conflict. Yet this result is based on the assumption that individual risks are independent and has thus limited relevance for real world catastrophic events. We extend Keeney’s result to dependent risks and derive the conditions under which more correlation and more equity between two risks induces a more catastrophic situation. We then generalize some of the results for multiple correlated risks.

Key words: risk equity, catastrophe aversion, correlation, dependence.
1 Introduction

The expected number of fatalities is perhaps the most common measure to assess and manage social risks. However, the expectation operation does not account for important dimensions of risk (Slovic et al., 1984). It neither reflects society’s preferences to avoid large scale accidents, nor does it capture concerns over inequalities in the distribution of risk across individuals (Bovens and Fleurbaey, 2012). Alternative criteria for managing public risks have therefore emerged. These criteria aim at limiting the maximum probable loss or the maximum individual risk, reflecting society’s anxiety to avoid the “bunching” of fatalities and its reluctance to accept risks that are unequally distributed across people.

In his seminal work, Ralph Keeney (1980) formally defined the concepts of risk equity and catastrophe aversion to capture both objectives. Risk equity is an *ex ante* concept corresponding to a preference for equalizing the probability of dying across agents. Catastrophe aversion, on the other hand, is an *ex post* concept corresponding to a preference for a mean-preserving concentration in the distribution of fatalities (Adler et al., 2014). Assuming independent risks, Keeney (1980) showed that the two concepts are always in conflict. Whenever one increases risk equity, the distribution of fatalities becomes more catastrophic and vice versa. This result is challenging as it highlights the conflict between two reasonable objectives of risk managers: limiting the risk burden to individuals and to society as a whole. It has received some attention in the operations research and management literature (e.g., Fishburn, 1984; Keeney and Winkler, 1985; Sarin, 1985; Fishburn and Straffin, 1989; Fishburn and Sarin, 1991, 1994, 1997; Gajdos et al., 2010), and more recently in the economics and social choice literature (Bommier and Zuber, 2008; Fleurbaey, 2010; Bovens and Fleurbaey, 2012; Adler et al., 2014).

In this paper, we examine an aspect of the problem that has so far been largely overlooked—the dependence structure of social risks. In today’s world, interdependent risks are the rule rather than the exception. This is particularly true for potentially catastrophic risks such as hurricanes, terrorist attacks, climate change or industrial accidents. We first show that the more correlated the risks faced by two agents are, the more catastrophic the distribution of fatalities is. We then derive the necessary and sufficient condition under which an equity-increasing risk transfer between two agents implies a more catastrophic distribution of risks. This condition pins down the effects of more risk equity on both the marginal distributions of the two risks and their correlation. We demonstrate that the condition always holds when a risk transfer has a positive effect on the correlation between the two risks.

Extending the analysis to more than two agents is challenging because pairwise correlations provide an insufficient statistic to map out the dependence structure of multiple risks (Embrechts et al., 2002). Nonetheless, we are able to generalize some of the results obtained for the two-agent case by imposing that, in a *N*-agent world, risk transfers between any two agents do not affect the dependence structure of the remaining *N* − 2 agents, or by weakening the notion of more catastrophic.
2 Catastrophic and correlated risks

2.1 Definitions and notations

Consider a population of \( i = 1, \ldots, N \) agents, each of whom faces an individual probability of dying \( p_i \in [0, 1] \). The risk of death is modeled as a Bernoulli random variable \( \tilde{x}_i \), which takes the value 1 (i.e., agent \( i \) dies) with probability \( p_i \), and 0 otherwise. We are interested in the distribution of fatalities:

\[
\tilde{d} := \sum_{i=1}^{N} \tilde{x}_i.
\]

As in Adler et al. (2014), we define a more catastrophic distribution of fatalities based on the definition of second-order stochastic dominance (Rothschild and Stiglitz, 1970).

**Definition 1.** A distribution of fatalities \( \tilde{d} \) is more catastrophic than another distribution \( \tilde{d}' \) iff for any concave function \( f \),

\[
E[f(\tilde{d})] \leq E[f(\tilde{d}')] \tag{1}
\]

Observe that, when \( f \) is linear, \( E[\tilde{d}] = E[\tilde{d}'] \). This implies that \( \tilde{d} \) is a mean-preserving spread of \( \tilde{d}' \). “Catastrophe avoidance” as defined by Keeney (1980, Theorem 2, p. 532) is a particular case of Definition 1, in which \( \tilde{d} \) and \( \tilde{d}' \) apply to binary risks with one safe outcome (i.e., an outcome which implies no fatality at all). Keeney’s analysis assumes that the social planner applies the axioms of expected utility to evaluate the number of fatalities occurring in scenarios with different probabilities. Accordingly, catastrophe aversion can be formally defined as a concave (social) vNM utility function \( f \).\(^1\) Now, let \( f(\tilde{d}) = -\tilde{d}^2 \). It follows immediately that a more catastrophic distribution must have a greater variance, but not the other way around. This leads to our second definition.

**Definition 2.** A distribution of fatalities \( \tilde{d} \) is more variable than another distribution \( \tilde{d}' \) iff \( \text{var}(\tilde{d}) \geq \text{var}(\tilde{d}') \).

Based on these two definitions, we will analyze the relationship between catastrophe aversion and risk equity. To begin with, we focus on two agents whose risks of death, \( \tilde{x}_1 \) and \( \tilde{x}_2 \), may be correlated. We introduce the probability space \( \Omega \), comprised of a finite (but potentially large) number of states \( S \), to describe the dependence structure of the two risks. For now, let us assume that \( \Omega \) has \( S := 8 \) equiprobable states \( (\omega_1, \ldots, \omega_8) \) and consider two agents, 1 and 2, who face the probability of dying \( p_1 \) and \( p_2 \), respectively.\(^2\)

Throughout the paper, we will make use of the following matrix notation to illustrate risky social situations.

\(^{1}\)It is, however, not so obvious that society should always display catastrophe aversion. See Rheinberger and Treich (2016) for an extensive discussion.

\(^{2}\)We use equiprobable states to ease the exposition of our examples. We emphasize, however, that the results also hold for unequal state probabilities as each state can be broken down into several equiprobable states.
Each of the $S := 8$ rows in this matrix corresponds to one possible state of the world. For $s = 1, \ldots, 8$, the $i$th value in row $s$ can be interpreted as $\tilde{x}_i(\omega_s)$, which is the value taken by the random variable $\tilde{x}_i$ in state $\omega_s$. In situation (A), there is only one state ($\omega_1$) in which the two agents die simultaneously. Each state in situation (A) has the same probability $1/S = \frac{1}{8}$ to occur. $\tilde{x}_i(\omega)$ can take two values: 1 with probability $p_1 := \Pr(\tilde{x}_1 = 1) = \frac{1}{2}$ (corresponding to the occurrence of a state in which the first agent dies) and 0 with probability $1 - p_1 = \frac{1}{2}$ (corresponding to a state in which the first agent survives). Likewise, the probability that the second agent dies is $p_2 := \Pr(\tilde{x}_2 = 1) = \frac{1}{4}$. Note that $\tilde{x}_2$ does not depend on the realizations of $\tilde{x}_1$ and vice versa: $\Pr(\tilde{x}_2 = 1|\tilde{x}_1 = 1) = \Pr(\tilde{x}_2 = 1)$ and $\Pr(\tilde{x}_1 = 1|\tilde{x}_2 = 1) = \Pr(\tilde{x}_1 = 1)$. In other words, the two risks are independent.

Based on the matrix notation, the computation of the distribution of fatalities $\tilde{d}_A := \tilde{x}_1 + \tilde{x}_2$ is straightforward. We only need to sum the values in each row to find that the probabilities of observing zero, one, and two fatalities are equal to $\pi_0 := \Pr(\tilde{d} = 0) = \frac{3}{8}$, $\pi_1 := \Pr(\tilde{d} = 1) = \frac{4}{8}$, and $\pi_2 := \Pr(\tilde{d} = 2) = \frac{1}{8}$, respectively. The corresponding distribution of fatalities is represented by the probability tree next to the matrix.

### 2.2 Correlation and the distribution of fatalities

In a two-agent world, there is a fundamental relationship between the distribution of fatalities and the correlation of the risks, which we capture by Proposition 1. The proposition assumes fixed marginal distributions (i.e., the parameters $p_1$ and $p_2$ are kept fixed). Thus, at this stage, there is no change in risk equity involved.

**Proposition 1.** Under $N = 2$, the four following statements are equivalent:

1. the probability of simultaneous fatalities increases;
2. the correlation between the individual risks increases;
3. the distribution of fatalities is more catastrophic (Definition 1);
4. the distribution of fatalities is more variable (Definition 2).
Proof. We first prove that the distribution becomes more catastrophic iff the probability of simultaneous fatalities, $\pi_2 := \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1)$, increases. For $N = 2$ agents, $\mathbb{E}[f(\tilde{d})] = \pi_0 f(0) + \pi_1 f(1) + \pi_2 f(2)$, where $\pi_i := \Pr(\tilde{d} = i)$ for $i = 0, 1, 2$. We know that $\mathbb{E}[\tilde{d}] = p_1 + p_2$, so that $\pi_1 + 2\pi_2 = p_1 + p_2$ (for $f(x) = x$) and $\pi_0 + \pi_1 + \pi_2 = 1$. Using these two equalities, we can express $\pi_0$ and $\pi_1$ as functions of $\pi_2$: $\mathbb{E}[f(\tilde{d})] = (1 - (p_1 + p_2 - 2\pi_2) - \pi_2)f(0) + (p_1 + p_2 - 2\pi_2)f(1) + \pi_2 f(2)$. This expression can be further simplified to

$$\mathbb{E}[f(\tilde{d})] = (1 - p_1 - p_2)f(0) + (p_1 + p_2)f(1) + \pi_2(f(0) - 2f(1) + f(2)).$$

By Jensen’s inequality we have $f(0) - 2f(1) + f(2) \leq 0$ for all $f$ concave. Therefore, $\mathbb{E}[f(\tilde{d})]$ decreases iff $\pi_2$ increases. Thus (iii)$\iff$(i).

Next, we turn to the joint probability of two random Bernoulli variables. We have

$$\pi_2 = \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \mathbb{E}[\tilde{x}_1 \tilde{x}_2] = p_1 p_2 + \rho \sqrt{p_1(1-p_1)} \sqrt{p_2(1-p_2)}$$

by the definition of the correlation $\rho$ coefficient between the two risks $\tilde{x}_1$ and $\tilde{x}_2$. Thus $\pi_2$ increases whenever $\rho$ increases: (i)$\iff$(ii). To conclude the proof, observe that

$$\text{var}(\tilde{x}_1 + \tilde{x}_2) = p_1(1-p_1) + p_2(1-p_2) + 2\rho \sqrt{p_1(1-p_1)} \sqrt{p_2(1-p_2)},$$

which increases iff correlation $\rho$ increases. Thus (ii)$\iff$(iv).

Let us provide an intuition for this result by modifying the introductory example (A). The distribution of fatalities can be more or less catastrophic depending only on the interaction between the two individual risks of death. In situation (B) below, we alter the dependence structure between $\tilde{x}_1$ and $\tilde{x}_2$ so that the occurrence of two simultaneous fatalities becomes most likely. By contrast, situation (C) illustrates a dependence structure in which two simultaneous fatalities are impossible:

\[
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

(B) \quad (C)
The risky situation depicted in (B) leads to the most catastrophic distribution of fatalities. This situation, in which the worst outcome becomes most likely, is known as the comonotonic dependence structure. By contrast, the risky situation (C) gives rise to the least catastrophic distribution of fatalities. This situation is also known as the antimonotonic dependence structure because outcomes are ordered in reverse order, thereby minimizing the probability to observe simultaneous fatalities. Indeed, $\tilde{d}_B$ is more catastrophic (and also more variable) than $\tilde{d}_C$.

The result in Proposition 1 is linked to Epstein and Tanny's (1980) concept of "generalized correlation". This concept characterizes the condition under which two random variables $\tilde{x}_1$ and $\tilde{x}_2$ are more correlated than two other random variables $\tilde{x}_1'$ and $\tilde{x}_2'$ (see also Tchen, 1980 and Wright, 1987). Epstein and Tanny show that this condition is formally equivalent to $E[u(\tilde{x}_1, \tilde{x}_2)] \leq E[u(\tilde{x}_1', \tilde{x}_2')]$ whenever the cross partial derivatives are nonpositive, i.e. $u_{12} \leq 0$. Now, take $u(x_1, x_2) = f(x_1 + x_2)$ so that $u_{12} \leq 0$ is equivalent to $f'' \leq 0$. This proves that an increasing "generalized correlation" between $\tilde{x}_1$ and $\tilde{x}_2$ is necessary and sufficient for obtaining a more catastrophic distribution $\tilde{x}_1 + \tilde{x}_2$, with $\tilde{x}_1$ and $\tilde{x}_2$ being Bernoulli random variables.

From Proposition 1 we know that the distribution of fatalities becomes most (least) catastrophic when the correlation between the two risks is maximized (minimized). In the case of two agents, it is always possible to fully identify the least and most catastrophic distribution of fatalities. This result is summarized in the following lemma, and will turn out to be useful later. In particular, we notice that the range of correlation between two risks depends on their marginal probabilities.

**Lemma 1.** The correlation $\rho$ between two Bernoulli random variables $\tilde{x}_1$ and $\tilde{x}_2$ with $p_1 > p_2$ is characterized by:

- $\rho \in \left[ -\frac{\sqrt{p_1 p_2}}{\sqrt{1-p_1} \sqrt{1-p_2}}, \frac{\sqrt{p_1 p_2}}{\sqrt{p_1} \sqrt{1-p_2}} \right]$, if $p_1 + p_2 \leq 1$
- $\rho \in \left[ -\frac{\sqrt{1-p_1} \sqrt{1-p_2}}{\sqrt{p_1 p_2}}, \frac{\sqrt{p_1 p_2}}{\sqrt{1-p_1} \sqrt{1-p_2}} \right]$, if $p_1 + p_2 > 1$
- $\rho = 0$, if $p_1 = 1$ (agent 1 is certain to die) or $p_2 = 0$ (agent 2 is certain to survive)

**Proof.** The original proof of Lemma 1 is due to Meilijson and Nadas (1979) and Tchen (1980). Here, we only provide a sketch of the proof. We start with minimum correlation, which is attained when the Bernoulli variables are antimonotonic, i.e. when the number
of simultaneous deaths is minimized. Specifically, we know that \( \pi_2 = \mathbb{E}[\tilde{x}_1 \tilde{x}_2] = \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \max(0, p_1 + p_2 - 1) \). There are two cases \( p_1 + p_2 \leq 1 \) and \( p_1 + p_2 > 1 \). In both cases, the minimum correlation is equal to

\[
\frac{\max(0, p_1 + p_2 - 1) - p_1 p_2}{\sqrt{p_1 (1 - p_1) p_2 (1 - p_2)}}.
\]

By considering these two cases separately and after some simplifications, one obtains the expressions of minimum correlation presented in the proposition.

The maximum correlation is obtained when the Bernoulli variables are comonotonic, i.e. when the number of simultaneous fatalities is maximized. In the two-agent world comonotonicity means that, in each state in which agent 2 dies, agent 1 dies as well. Because the maximum probability of simultaneous fatalities is equal to \( \Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1) = \min(p_1, p_2) = p_2 \), their maximum correlation equals

\[
\frac{p_2 - p_1 p_2}{\sqrt{p_1 (1 - p_1) p_2 (1 - p_2)}} = \frac{\sqrt{p_2 \sqrt{1 - p_1}}}{\sqrt{p_1 \sqrt{1 - p_2}}},
\]

One practical remark on Lemma 1 seems in order. The natural bounds of the correlation interval are only attained in the special case where \( p_1 = p_2 = \frac{1}{2} \). This implies that for most binary risks the interval of attainable correlation is strictly narrower than \( \rho \in [-1, 1] \).

### 3 Risk equity and its implications

Following Keeney (1980) and the subsequent literature cited in the introduction, we define risk equity based on a “Pigou-Dalton” transfer in risk: a non-leaky transfer of probability mass from a more exposed to a less exposed individual, so that the transfer does not reverse the ranking of the two individuals in terms of their probability to die.\(^3\)

**Definition 3.** A distribution of fatalities \( \tilde{d} \) is more equitable than another distribution \( \tilde{d}' \) iff a Pigou-Dalton transfer in risk \( \delta \) from a more exposed agent 1 to a less exposed agent 2 would reduce the exposure of agent 1 and raise the exposure of agent 2 without switching their ranking in terms of absolute exposure and without changing other individuals’ exposure. Formally, the probabilities of dying before the transfer are \( p_1 \) and \( p_2 \), with \( p_1 > p_2 \); after the transfer, the probabilities of dying are \( p'_1 = p_1 - \delta \) and \( p'_2 = p_2 + \delta \), where \( 0 \leq \delta \leq \frac{p_1 - p_2}{2} \).

Accordingly, a Pigou-Dalton transfer in risk always decreases the “gap” in risk exposure between the agents 1 and 2 (Keeney, 1980; Rothschild and Stiglitz, 1973; Adler

\(^3\)Interestingly, Cox (2012) argues that—unlike income—mortality risk is not fungible and cannot be transferred from one individual to another. Our definition does, however, not imply a physical transfer of risk from one individual to another. It merely implies the existence of different policy options under which two individuals face different risk exposures.
et al., 2014). This definition is intuitive, but also general since any mean-preserving contraction in the distribution of individual risks within society can be obtained through a series of Pigou-Dalton transfers. Our first objective is to extend Keeney’s result to our more general definition of catastrophic risk. Proposition 2 asserts that, under the assumption of independent risks before and after the transfer, more risk equity always implies a more catastrophic distribution of fatalities.

**Proposition 2.** Assume that the risks faced by \( N \) agents are independent. In that case, any Pigou-Dalton transfer in risk between two agents leads to a more catastrophic distribution of fatalities, if the risks to the \( N \) agents remain independent after the transfer.

**Proof.** Under the assumption of independent risks it is sufficient to focus on the risks affected by the transfer. Consider for example a change in the risks to agents 1 and 2, and denote \( \tilde{y} := \tilde{x}_3 + \ldots + \tilde{x}_N. \) We define \( E[f(\tilde{d})] = E[f_1(\tilde{x}_1 + \tilde{x}_2)], \) with \( f_1(x) = E[f(x + \tilde{y})]. \) Since the function \( f_1 \) is concave iff \( f \) is concave under risk independence, the presence of \( N - 2 \) independent agents does not affect the comparative statics analysis. From Proposition 1 and the equivalence between a more catastrophic and a more variable distribution of fatalities, we know that it suffices to show that a Pigou-Dalton transfer in risk between agents 1 and 2 increases the variance. This is always true because

\[
\text{var}[\tilde{d}'] - \text{var}[\tilde{d}] = [(p_1 - \delta)(1 - p_1 + \delta) + (p_2 + \delta)(1 - p_2 - \delta)] - [p_1(1 - p_1) + p_2(1 - p_2)],
\]

which is positive for any \( \delta \leq \frac{p_1 - p_2}{2} \) (which is exactly the condition needed for the Pigou-Dalton transfer).

\[\blacksquare\]

### 3.1 Two agents facing dependent risks

Our next objective is to extend the result in Proposition 2 above to the case of dependent risks. To illustrate the complexity that arises from the interaction between the two risks, we start from example (A) presented in section 2.1 and analyze three possible risk transfers and their respective effect on the distribution of fatalities. Remember that in situation (A) the risks of the two agents are independent. Here, and in contrast to Keeney (Proposition 2), we do not assume that the risks remain independent after the Pigou-Dalton transfer. Consider the following situations labeled (D), (E), and (F), respectively. (Risks after a transfer of \( \delta = \frac{1}{2} \) are denoted by \( \tilde{x}_1' \) and \( \tilde{x}_2' \).) In all three situations, the agents face the same probability to die (\( p_1' := p_1 - \delta = p_2' := p_2 + \delta = \frac{3}{4} \)). However, in none of the situations the two risks are independent and, consequently, Proposition 2 no longer applies.
The new correlation $\rho'$ between the agents' risks can be computed as follows:

$$
\rho' := \text{corr}(\tilde{x}_1', \tilde{x}_2') = \frac{\text{E}[\tilde{x}_1' \tilde{x}_2'] - (p_1 - \delta)(p_2 + \delta)}{\sqrt{(p_1 - \delta)(1 - p_1 + \delta) \sqrt{(p_2 + \delta)(1 - p_2 - \delta)}}}.
$$

For the above situations, we have: $\rho'_D \approx -0.06$, $\rho'_E = -0.6$, and $\rho'_F = 1$, respectively.

The distribution of fatalities $\tilde{d}' := \tilde{x}_1' + \tilde{x}_2'$ after each of the feasible Pigou-Dalton transfers is fully characterized by the corresponding probability trees. Compared to situation (A), the distribution of fatalities becomes strictly more catastrophic in situation (F), strictly less catastrophic in situation (E), and is identical in situation (D).

Situations (A) to (F) make it clear that the change in the distribution of fatalities is governed by two sources: (i) the effect of the Pigou-Dalton transfer in risk on the marginal distributions, and (ii) the change in correlation induced by the transfer. Both changes have an impact on how catastrophic the distribution of fatalities is. Situations (B) and (C) have illustrated that an increase (or decrease) in catastrophic risk might be caused by a change in correlation only. Situations (D) to (F) illustrate that an increase (or decrease) in catastrophic risk might also be due to a simultaneous change in correlation and risk equity.

The comparison of (A) with (D) is particularly revealing as (D) is obtained from (A) by switching a “1” and a “0” in a single row, and by relabeling the states. The distributions of fatalities are fully determined by the number of “1’s” in each row. Therefore, the two distributions must be identical before and after the transfer. Indeed, $\tilde{d}'_A$ and $\tilde{d}'_D$ have the same distribution. In other words, the induced change in the
marginal distributions of individual risks $\tilde{x}_i$ is “counteracted” by its negative impact on the correlation, which keeps the sum of $\tilde{x}$’s identically distributed. This insight underlines that it is not straightforward to extend Proposition 2, and there is no hope to generically sign the comparative statics analysis of any possible risk transfer without precise restrictions on the correlation.⁴

3.2 The necessary and sufficient condition

In this section we identify the necessary and sufficient condition under which a Pigou-Dalton transfer in risk between two agents leads to a more catastrophic distribution of fatalities. The condition includes the special case in which the risks have identical correlation before and after the transfer as in the previous examples. Yet we also provide a more general analysis, in which we allow changes between the correlation before and after the transfer. This degree of generality is important for many real world applications. We demonstrate that a Pigou-Dalton transfer in risk makes the distribution of fatalities more catastrophic whenever the correlation after the transfer is larger than a specified threshold.

Proposition 3. Assume $N = 2$ and $p_1 > p_2$. Let $\rho$ denote the correlation between the initial risks $\tilde{x}_1$ and $\tilde{x}_2$. After a Pigou-Dalton transfer in risk $\delta \in [0, \frac{p_1-p_2}{2}]$, the distribution of fatalities becomes more catastrophic iff the correlation $\rho'$ between $\tilde{x}'_1$ and $\tilde{x}'_2$ is larger than the critical level of correlation $\rho^*$, i.e.

$$\rho' \geq \rho^* := \frac{\delta(p_2-p_1+\delta) + \rho \sqrt{p_1(1-p_1)} \sqrt{p_2(1-p_2)}}{\sqrt{1-p_1+\delta} \sqrt{p_1-\delta} \sqrt{1-p_2-\delta}}.$$  (1)

Proof. We first extend a result in Proposition 1 showing that even when the marginals $p_1$ and $p_2$ are not kept fixed before and after the risk transfer, an increase in the probability of simultaneous deaths is equivalent to a more catastrophic distribution of fatalities. Let $\pi'_2 = \pi_2 + \gamma \geq 0$. Keeping the number of expected fatalities constant, i.e. $\pi'_1 + 2\pi'_2 = \pi_1 + 2\pi_2$, we have $\pi'_1 = \pi_1 - 2\gamma \geq 0$ and in turn $\pi'_0 = \pi_0 + \gamma$, because the sum of probabilities must equal one. Therefore, the distribution is more catastrophic iff $\gamma \geq 0$, and thus $\pi'_2 \geq \pi_2$.

Next, we compute the respective probabilities of simultaneous deaths before and after the risk transfer. We find that

$$\pi_2 = E[\tilde{x}_1 \tilde{x}_2] = p_1p_2 + \rho \sqrt{p_1(1-p_1)} \sqrt{p_2(1-p_2)},$$  (2)

and

$$\pi'_2 = E[\tilde{x}'_1 \tilde{x}'_2]$$
$$= (p_1 - \delta)(p_2 + \delta) + \rho' \sqrt{(p_1 - \delta)(1-p_1 + \delta) \sqrt{(p_2 + \delta)(1-p_2 - \delta)}}.$$  (3)

⁴For $S := 8$ states ($\omega_1, \ldots, \omega_8$), it is impossible to find a situation in which the correlation between $\tilde{x}'_1$ and $\tilde{x}'_2$ is equal to 0 (i.e., in which the risks are still independent after the risk transfer) and for which the result of Keeney (Proposition 2) would hence hold. It is, however, possible to construct such a situation by invoking more states of the world. We provide one such example in Appendix A.1.
Therefore, \( \pi'_2 \geq \pi_2 \) iff the correlation \( \rho' \) is sufficiently high (i.e., iff condition (1) is satisfied).

As stated above, a Pigou-Dalton transfer in risk may have two distinct effects on the distribution of fatalities: i) through the change in the marginal distributions and ii) through a change in the correlation between the two risks. Condition (1) in Proposition 3 depends on both effects, and it seems useful to think about them separately. The effect on the marginal distributions corresponds to a change in \( \delta \), keeping the correlation structure fixed: \( \rho' = \rho \). The effect on the dependence structure corresponds to a change in correlation from \( \rho \) to \( \rho' \) assuming no changes in the marginal distributions (i.e., \( \delta = 0 \)). However, it would be fallacious to separate the two effects as the change in correlation is bounded (see Lemma 1), and the range over which it is defined depends on the marginal distributions.\(^5\) In other words, the range of attainable correlation between \( \tilde{x}_1' \) and \( \tilde{x}_2' \) is a function of \( \delta \), because the correlation \( \rho' \) between two risks with respective probabilities \( p_1-\delta \) and \( p_2+\delta \) cannot be larger than

\[
\rho_{\text{max}}(\delta) := \frac{\sqrt{p_2+\delta} \sqrt{1-p_1+\delta}}{\sqrt{p_1-\delta} \sqrt{1-p_2-\delta}}. \tag{4}
\]

The minimum correlation depends on whether \( (p_1-\delta) + (p_2+\delta) = p_1 + p_2 \) is larger than 1 or not and is also defined by Lemma 1. Furthermore, \( \rho_{\text{max}}(0) \) equals the maximum correlation \( \rho \) between the initial risks if \( \delta = 0 \).

Figure 1 displays the comparative statics analysis for a numerical example where the two effects are simultaneously at work. The grey-shaded areas in the four panels of Figure 1 represent the admissible range for the transfer \( \delta \) (on the x-axis) and the correlation \( \rho' \) (on the y-axis) parameters such that the distribution of fatalities is more catastrophic after the Pigou-Dalton transfer (the example assumes that \( p_1 = 0.8 \), \( p_2 = 0.3 \), \( \delta \in [0, \frac{p_1-p_2}{2}] \), and considers different initial values of \( \rho \)).

Specifically, Figure 1 illustrates four situations in which the initial correlation between the two agents’ risks is equal to \( \rho = \rho_{\text{min}}(0) = -0.76 \) (Panel A), \( \rho = -0.5 \) (Panel B), \( \rho = 0 \) (Panel C), and \( \rho = \rho_{\text{max}}(0) = 0.33 \) (Panel D), respectively. Each panel plots the critical level of correlation \( \rho^* \) as well as the attainable minimum and maximum correlation (4) as a function of \( \delta \) using the relationships of Lemma 1.

Several special cases of Proposition 3 are worth to be discussed in more detail. We summarize them in Proposition 4.

**Proposition 4.** Assume \( N = 2 \) and \( p_1 > p_2 \). In the following special cases, the distribution of fatalities becomes more catastrophic after a Pigou-Dalton transfer in risk:

(i) \( \rho' \geq \rho \) and \( \delta > 0 \) (which includes Keeney’s (1980) result for \( \rho' = \rho = 0 \), and fixed correlation \( \rho' = \rho \), as special cases);

(ii) \( \delta = 0 \) and \( \rho' > \rho \) (no change in the marginal distribution of fatalities, but an increase in correlation);

\(^5\)This implies that the comparative statics analysis of a change in the risk transfer \( \delta \), assuming that the correlation \( \rho \) is kept constant, can be fallacious.
(iii) \( \rho \) is equal to the minimum correlation (as computed in Lemma 1) between the two risks before the Pigou-Dalton transfer;

(iv) \( \rho' \) is equal to the maximum correlation (as computed in Lemma 1) between the two risks after the Pigou-Dalton transfer;

(v) \( p_1 = 1 \) (agent 1 is certain to die) or \( p_2 = 0 \) (agent 2 is certain to survive).

Proof. We start from the expressions of \( \pi_2 \) and \( \pi'_2 \) given by Eqs. (2)-(3) and show that \( \pi'_2 \geq \pi_2 \) in each of the above statements (i)-(v).

In order to prove (i), note that by definition a Pigou-Dalton transfer imposes \( \delta \leq \frac{p_1 - p_2}{2} \), so that \( p_1p_2 \leq (p_1 - \delta)(p_2 + \delta) \) and \( \sqrt{1 - p_1 - \delta} \sqrt{p_2 + \delta} > \sqrt{p_1 \sqrt{1 - p_1} - p_2 \sqrt{1 - p_2}} \). For \( \rho' \geq \rho \geq 0 \), the proof of (i) follows directly from the expressions of \( \pi_2 \) and \( \pi'_2 \) in Eqs. (2)-(3) and \( \pi'_2 \geq \pi_2 \). For \( \rho \leq \rho' < 0 \), the result still holds but the proof is longer and therefore relegated to Appendix A.2.
The proof of (ii) follows directly from Proposition 1. This case isolates the effect of correlation.

In order to prove (iii), observe that there are two cases for which the correlation is minimum (see Lemma 1). If \( p_1 + p_2 \leq 1 \), then \( \pi_2 = 0 \) is obtained after replacing the correlation in Eq. (2) by \( -\sqrt{p_1 p_2} \). Thus, \( \pi_2' \geq 0 = \pi_2 \). If \( p_1 + p_2 > 1 \), then the expression of \( \pi_2 \) can be simplified to \( p_1 + p_2 - 1 \). We apply Lemma 1 to the marginals \( p_1 - \delta \) and \( p_2 + \delta \) to compute the minimum correlation \( \rho_{\text{min}} \). From \( \pi_2' \geq (p_1 - \delta)(p_2 + \delta) + \rho_{\text{min}} \sqrt{1 - p_1 \sqrt{1 - p_2}} \), and replacing \( \rho_{\text{min}} \) by its expression, we find \( \pi_2' \geq p_1 + p_2 - 1 = \pi_2 \).

The proof of (iv) is almost identical. After replacing \( \rho' \) in Eq. (3) by the maximum correlation, we find that \( \pi_2' = p_2 + \delta \). However \( \pi_2 = p_1 p_2 + \rho \sqrt{1 - p_1 \sqrt{1 - p_2}} \leq p_1 p_2 + \rho_{\text{max}} \sqrt{1 - p_1 \sqrt{1 - p_2}} \). Some simplifications yield \( \pi_2 \leq p_2 < p_2 + \delta = \pi_2' \).

Finally, we prove (v) as follows. First, observe that for \( p_1 = 1 \) we have \( \rho^* := \frac{\delta (p_2 - 1 - \delta)}{\sqrt{\delta (1 - \delta)} \sqrt{(p_2 + \delta)(1 - p_2 - \delta)}} \), which equals the minimum bound identified by Lemma 1 (for \( p_1 + p_2 > 1 \)). Second, observe that for \( p_2 = 0, \rho^* = \frac{\delta (-p_1 + \delta)}{\sqrt{(1 - p_1 + \delta)(p_1 - \delta)} \sqrt{\delta (1 - \delta)}} \), which equals the minimum bound identified by Lemma 1 (for \( p_1 + p_2 \leq 1 \)). Thus, we always obtain \( \rho' \geq \rho^* \) for these two special cases. 

Statement (i) of Proposition 4 is apparent in all of the four panels in Figure 1: the horizontal line representing the level \( \rho' = \rho \) always belongs to the grey-shaded area where the distribution of fatalities becomes more catastrophic. Statement (ii) follows immediately from the inspection of the \( \rho \)-values at \( \delta = 0 \). Statement (iii) corresponds to Panel A in Figure 1, for which the correlation between the initial risks is minimum. Statement (iv) is, again, apparent in all four panels as the dashed line that represents the maximum correlation for \( \rho' \) always belongs to the grey-shaded area, in which the distribution of fatalities is more catastrophic. Statement (v) illustrates a Pigou-Dalton transfer between two agents in which the correlation between the initial risks is equal to 0, which is also the minimum attainable correlation.

4 Generalization to more than two agents

We already observed in the two-agent world that the distribution of fatalities can become more or less catastrophic when either the correlation or the marginal distributions are altered by a Pigou-Dalton transfer in risk. In this section, we parallel the previous discussion for \( N > 2 \) agents. As in section 2.2, we first discuss whether the distribution of fatalities becomes more (or less) catastrophic when only the dependence structure changes. We then extend section 3 and look at the effect of a change in marginal distributions of two risks in the presence of \( N - 2 \) other agents. Under the assumption that the risk dependence structure of the \( N - 2 \) other agents is kept fixed, we show that the results obtained for \( N = 2 \) still hold. We then relax the constraint on the risk dependence structure and, instead, make assumptions about pairwise correlations. As is well known, pairwise correlations alone do not provide sufficient information to
pin down the dependence structure of the $\tilde{x}_1, ..., \tilde{x}_N$ risks. We can, however, show that the distribution of fatalities becomes more variable after a Pigou-Dalton transfer with uncorrelated risks. At the end of section 4, we derive more general sufficiency conditions under which the distribution of fatalities becomes more variable after a Pigou-Dalton transfer in risk.

4.1 Extremal dependence and the distribution of fatalities

In situations with $N > 2$ agents it is not obvious how one should compare distributions of fatalities against each other as it is unclear how the dependence structure between $N$ risks should be defined. Even for Bernoulli random variables, the dependence structure involves more than the pairwise correlation coefficients $\rho_{ij}$ (as in section 2.2).\footnote{See, for instance, an example of uncorrelated risks that are not independent, situations (J) and (K) presented in section 4.3.} Without restricting the dependence structure of the $N$-agents’ risks, we can only derive results for the two extreme cases of comonotonic and antimonotonic dependence. These two extreme dependence structures are defined as follows.

**Definition 4.** Consider $i = 1, 2, ..., N$ agents with $p_i \in (0, 1)$ and let $\tilde{d} = \tilde{x}_1 + ... + \tilde{x}_N$ be the corresponding distribution of fatalities. Let then $\tilde{d}^c$ denote the comonotonic dependence structure (also known as the maximum correlation among the risks). Moreover, let $\tilde{d}^a$ denote the antimonotonic dependence structure implying that in all states either $M$ or $M + 1$ deaths occur. Formally, $M$ is the integer number such that the expected number of fatalities is $\mu := p_1 + p_2 + ... + p_N \in [M, M + 1]$. The distribution of fatalities $\tilde{d}^a$ thus takes one of two values: $M$ with probability $p_M = M + 1 - \mu$, or $M + 1$ with probability $1 - p_M = \mu - M$.

Note that the expression of $p_M$ is computed such that the expected number of fatalities is preserved; i.e., $p_M M + (1 - p_M)(M + 1) = \mu$. Based on this definition, we obtain the following intuitive result.

**Proposition 5.** The distribution of fatalities $\tilde{d}$ is always less catastrophic than $\tilde{d}^c$, and always more catastrophic than $\tilde{d}^a$. Namely, for all $f$ concave and all possible distributions of fatalities $\tilde{d}$:

$$
E[f(\tilde{d}^a)] \leq E[f(\tilde{d})] \leq E[f(\tilde{d}^c)] = f(M)p_M + f(M+1)p_{M+1}.
$$

Proposition 5 is closely related to Lemma 1 as it defines an admissible range of dependence for $N > 2$ agents (a detailed proof is provided in Appendix A.3). The most and least catastrophic distributions of fatalities, $\tilde{d}^c$ and $\tilde{d}^a$, are obtained when “correlation” is maximized and minimized, respectively.

Meilijson and Nadas (1979) proved that maximum correlation is obtained whenever risks are comonotonic or, in non-technical terms, concentrated to specific states of the world.

The least catastrophic distribution of fatalities is obtained by a generalization of the negative dependence structure in $N$ dimensions. This generalization is far from
trivial, however. Take the example of three risks—\(x\), \(y\), and \(z\)—and assume that \(x\) is negatively correlated with \(y\) and also negatively correlated with \(z\); then, by definition, \(y\) and \(z\) must be positively correlated.

This simple example highlights that it is not straightforward to define what it means that three variables are negatively correlated. Puccetti and Rüschendorf (2012) recently proposed a rearrangement algorithm, which approximates the antimonotonic dependence structure of high-dimensional problems.\(^7\) In the particular case of Proposition 5, all risks are Bernoulli distributed and the least catastrophic distribution of fatalities is therefore explicitly known (see Bernard et al. (2016) and the proof in Appendix A.3).

Let us further illustrate Proposition 5. Situations (\(G\)) to (\(I\)) have the same marginal distributions for \(x_1\), \(x_2\), and \(x_3\) (\(p_1 = 1/2\), \(p_2 = 1/4\), and \(p_3 = 1/2\)), but they differ in terms of their dependence structure.

\[
\begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

(\(G\)) (\(H\)) (\(I\))

Situation (\(G\)) is a situation with uncorrelated risks. Specifically, all pairs \(\{x_1, x_2\}\), \(\{x_1, x_3\}\), and \(\{x_2, x_3\}\) have pairwise zero correlation (\(\rho_{12} = \rho_{13} = \rho_{23} = 0\)). Situation (\(H\)) gives rise to the most catastrophic distribution of fatalities \(\tilde{d}_H\), and situation (\(I\)) to the least catastrophic distribution of fatalities \(\tilde{d}_I\). The expected number of fatalities is in all three situations \(\mu = E[\tilde{d}] = 1.25\), but the range of possible outcomes differs across

\(^7\)The rearrangement algorithm is based on the following idea. Denote the distribution of fatalities aggregated over all but one agent, say agent \(i\), by \(\tilde{d}_{-i} := \sum_{j \neq i} \tilde{x}_j\). The distribution of fatalities becomes less variable iff the correlation \(\rho_i\) between \(\tilde{d}_{-i}\) and \(\tilde{x}_i\) decreases (for any agent \(i = 1, 2, ..., N\)). This result follows from the fact that the variability of the distribution of fatalities can be expressed as

\[
\text{var}(\tilde{d}) = \text{var}(\tilde{x}_i + \tilde{d}_{-i}) = \text{var}(\tilde{x}_i) + \text{var}(\tilde{d}_{-i}) + 2\rho_i \sqrt{\text{var}(\tilde{d}_{-i}) \text{var}(\tilde{x}_i)},
\]

in which only \(\rho_i\) is affected.
the situations. In particular, \( d_i^\theta \) takes on only two values: one death with probability \( p_M = 1 + 1 - 1.25 = 0.75 \) and two deaths with probability \( 1 - p_M = 0.25 \). Also observe that the least catastrophic distribution of fatalities is such that each individual risk \( \tilde{x}_i \) is in reverse order with \( \tilde{d}_{-i} := \sum_{j \neq i} \tilde{x}_j \) (i.e., with the distribution of fatalities over all but agent \( i \)). In other words, the states wherein \( \tilde{x}_i = 1 \) correspond to the states of the smallest value of \( \tilde{d}_{-i} \) are attained. (As outlined in footnote 7, this is necessary to achieve minimum correlation.) Lastly, note that the variability of the distributions of fatalities largely differs across the situations: \( \text{var}(\tilde{d}_G) = 11/16 \) in \( (G) \), \( \text{var}(\tilde{d}_H) = 27/16 \) in \( (H) \), and \( \text{var}(\tilde{d}_I) = 3/16 \) in \( (I) \). By definition, the latter two variance terms are the maximum and minimum variance, respectively.

The result in Proposition 5 can be interpreted as a mild generalization of Proposition 1 as it indicates that the equivalence result between an increase in dependence and a more catastrophic distribution holds under \( N > 2 \) for the two most extreme distributions of fatalities. However, the result is not general because it does not characterize the effect of “more dependence”. Moreover, we can easily show that another equivalence result of Proposition 1 fails. It concerns the equivalence between an increasing probability of simultaneous fatalities and a more catastrophic distribution. A simple counterexample with \( N = 3 \) suffices to show this. Define \( \tilde{d} \) by \( (\pi_0, \pi_1, \pi_2, \pi_3) = (0, 3/5, 0, 2/5) \) and \( \tilde{d}' \) by \( (\pi_0, \pi_1, \pi_2, \pi_3) = (1/5, 0, 3/5, 1/5) \). Both distributions have the same mean (i.e., 9/5), but \( \tilde{d} \) implies a higher probability of simultaneous fatalities \( \pi_3 \) (i.e., 2/5 > 1/5). Nevertheless, \( \tilde{d} \) has a lower probability that at least two fatalities occur (i.e., 2/5 < 3/5). There exist (at least) two concave functions that do not imply the same order: \( \mathbb{E}(-\max(\tilde{d} - 1, 0)) = -0.8 > \mathbb{E}(-\max(\tilde{d}' - 1, 0)) = -1 \) and \( \mathbb{E}(-\max(\tilde{d} - 2, 0)) = -0.4 < \mathbb{E}(-\max(\tilde{d}' - 2, 0)) = -0.2 \). Therefore, \( \tilde{d} \) cannot be more catastrophic than \( \tilde{d}' \).

4.2 Pigou-Dalton transfers with fixed risk dependence

We now examine the impact of a Pigou-Dalton transfer in risk between two agents in an economy consisting of \( N > 2 \) agents. Remember that Keeney (1980)'s result considers \( N \) agents but assumes risk independence among the \( N \) agents. In practice, independent risks are often implausible. We now extend Keeney’s result assuming fixed risk dependence among the \( N - 2 \) other agents, who are not involved in the Pigou-Dalton transfer.

**Proposition 6.** Let \( N > 2 \), and consider a Pigou Dalton transfer in risk between agents 1 and 2. Assume that this transfer does not affect the risk dependence among the other agents in the following sense: only the probabilities that either agent 1 or 2 dies and that both agents 1 and 2 die simultaneously may be altered by the risk transfer. Then, the result given in Proposition 3 still holds: the distribution of fatalities becomes more catastrophic iff \( \rho' \geq \rho^* \) where \( \rho' \) is the correlation between agents 1 and 2’s risks after the risk transfer and \( \rho^* \) is the critical level of correlation given in Equation (1).

**Proof.** The following notation is useful to demonstrate this result. Let \( \Theta \) be a subset of \( \{1, 2, ..., N\} \) agents and \( p_\Theta \) be the probability that exactly this subset of agents die.
The probability that agent $i$ dies can then be rewritten as

$$p_i = p_{(i)} + \sum_{k_1 \neq i} p_{(i,k_1)} + \sum_{k_1,k_2 \neq i} p_{(i,k_1,k_2)} + ... + p_{(1,2,...,N)} \tag{5}$$

because agent $i$ can die alone or together with $k = 1, ..., N - 1$ other agents.

Without loss of generality, we show the result for $N = 3$. Using Definition (5), we have:

$$p_1 = p_{(1)} + p_{(1,2)} + p_{(1,3)} + p_{(1,2,3)}$$
$$p_2 = p_{(2)} + p_{(1,2)} + p_{(2,3)} + p_{(1,2,3)}$$
$$p'_1 = p'_{(1)} + p'_{(1,2)} + p'_{(1,3)} + p'_{(1,2,3)}$$
$$p'_2 = p'_{(2)} + p'_{(1,2)} + p'_{(2,3)} + p'_{(1,2,3)}.$$

Let us now apply a Pigou Dalton transfer in risk: $p'_1 = p_1 - \delta$ and $p'_2 = p_2 + \delta$, where $0 \leq \delta \leq \frac{p_1 - p_2}{2}$. Because of the assumption of fixed risk dependence, the Pigou-Dalton transfer in risk between agents 1 and 2 can only affect $p_{(1)}, p_{(2)}$ and $p_{(1,2)}$. Let the probability that agents 1 and 2 die simultaneously become: $p'_{(1,2)} = p_{(1,2)} + \gamma$. This leads to $p_1 - \delta = p'_{(1)} + p_{(1,2)} + \gamma + p'_{(1,3)} + p'_{(1,2,3)}$, which using $p_{(1,3)} = p'_{(1,3)}$ and $p_{(1,2,3)} = p'_{(1,2,3)}$ due to fixed risk dependence, implies

$$p'_{(1)} = p_{(1)} - \delta - \gamma.$$

Similarly, we have $p_2 + \delta = p'_{(2)} + p_{(1,2)} + \gamma + p'_{(2,3)} + p'_{(1,2,3)}$ which using again $p_{(2,3)} = p'_{(2,3)}$ and $p_{(1,2,3)} = p'_{(1,2,3)}$ implies

$$p'_{(2)} = p_{(2)} + \delta - \gamma.$$

Therefore, the probability of exactly one death after the Pigou-Dalton transfer is $\pi'_1 = p'_{(1)} + p'_{(2)} + p'_{(3)}$, which using $p'_{(3)} = p_{(3)}$, implies

$$\pi'_1 = \pi_1 - 2\gamma.$$

Similarly, the probability of exactly two deaths is $\pi'_2 = p'_{(1,2)} + p'_{(1,3)} + p'_{(2,3)}$, which using $p'_{(1,3)} = p_{(1,3)}$ and $p'_{(1,2)} = p_{(1,2)} + \gamma$, implies

$$\pi'_2 = \pi_2 + \gamma.$$

Finally, the probability that all $N = 3$ agents die simultaneously does not change due to fixed risk dependence regarding agent 3’s risk:

$$\pi'_3 = p'_{(1,2,3)} = p_{(1,2,3)} = \pi_3.$$
This further implies that the probability that nobody dies is equal to

\[ \pi'_0 = 1 - \pi'_1 - \pi'_2 - \pi'_3 \]
\[ = 1 - (\pi_1 - 2\gamma) - (\pi_2 + \gamma) - \pi_3 \]
\[ = \pi_0 + \gamma. \]

The expressions above for \( \pi'_0, \pi'_1, \pi'_2 \) and \( \pi'_3 \) permit to conclude that the distribution of fatalities becomes more catastrophic after the transfer in risk iff \( \gamma \geq 0 \), namely iff \( \pi'_2 \geq \pi_2 \). We may then follow the proof of Proposition 3 to demonstrate the result. The proof for \( N > 3 \) is analogous. □

Proposition 6 demonstrates that if the transfer in risk does not affect the risk dependence structure of the \( k \) other agents, Keeney’s result carries over to \( N > 2 \) agents provided it holds for the agents 1 and 2 (case \( N = 2 \)). For example, this last condition is met if the correlation between the two agents stays the same, i.e. \( \rho = \rho' \), as shown in Proposition 4.

Another special case is to assume a fixed risk dependence among all \( N \) agents, which implies fixed pairwise correlations and, therefore, \( \rho = \rho' \). As an immediate corollary of Proposition 6, Keeney’s result holds whenever the risk dependence among all \( N \) agents is kept fixed.

One may argue that keeping fixed risk dependence among the agents not involved in the Pigou-Dalton transfer is a very strong assumption. However, we show in the rest of the paper that if we relax this constraint we cannot extend Keeney’s result and one may not even be able to conclude about whether the resulting distribution of fatalities is more variable.

### 4.3 Pigou-Dalton transfers with uncorrelated risks

In the rest of the paper, we will relax the constraint on the risk dependence structure, and make instead assumptions about pairwise correlations. Even though it is well known that pairwise correlations provide an insufficient statistic to map out the dependence structure of multiple risks (Embrechts et al., 2002), the industry often relies on correlations to measure dependence. Many economic models build on factor model, which are determined by regressions and correlation coefficients.

In this section, we first start with a striking negative result. Indeed, we show that Keeney’s result does not even hold when all the \( N \) risks are uncorrelated. Again, we make use of a simple example. Consider situations \((J)\) and \((K)\), below. The two situations consist of \( S := 16 \) states of the world. In each situation, three agents are faced with the risks \( \tilde{x}_1, \tilde{x}_2, \) and \( \tilde{x}_3 \), whose pairwise correlation coefficients are equal to zero before \((J)\) and after the Pigou-Dalton transfer \((K)\). One can easily verify that

\[ \text{corr}(\tilde{x}_1, \tilde{x}_2) = \text{corr}(\tilde{x}_1, \tilde{x}_3) = \text{corr}(\tilde{x}_2, \tilde{x}_3) = 0 \]
\[ \text{corr}(\tilde{x}'_1, \tilde{x}'_2) = \text{corr}(\tilde{x}'_1, \tilde{x}_3) = \text{corr}(\tilde{x}'_2, \tilde{x}_3) = 0. \]

The distributions of fatalities, \( \tilde{d}_J \) and \( \tilde{d}'_K \), are again depicted as trees.
Observe that the variance increases from \( \text{var}(\tilde{d}_J) = \frac{5}{8} \) to \( \text{var}(\tilde{d}_K) = \frac{6}{8} \) due to the Pigou-Dalton transfer between agent 1 and 2, demonstrating that the post-transfer distribution of risk is more variable. However, \( \tilde{d}_K \) is not more catastrophic. Consider the concave function \( f(x) = -\max(x - \psi, 0) \). For \( \psi = 2.5 \), \( E[f(\tilde{d}_K)] = 0 > E[f(\tilde{d}_J)] \). Therefore, the Pigou-Dalton transfer results in an increased variability, but not in more catastrophic risk.

Situations \((J)\) and \((K)\) indicate that pairwise zero-correlation is insufficient to maintain Keeney’s result when risks are dependent. Of course, pairwise zero-correlation does not imply independence. In the above example, \( \tilde{x}_1, \tilde{x}_2, \) and \( \tilde{x}_3 \) are not independent:

\[
\Pr(\tilde{x}_1 = 1, \tilde{x}_2 = 1, \tilde{x}_3 = 1) = \frac{2}{16} \neq p_1p_2p_3 = \frac{3}{32}.
\]

Neither are \( \tilde{x}_1', \tilde{x}_2', \) and \( \tilde{x}_3' \):

\[
\Pr(\tilde{x}_1' = 1, \tilde{x}_2' = 1, \tilde{x}_3' = 1) = 0 \neq p_1'p_2'p_3 = (p_1 - \delta)(p_2 + \delta)p_3 = \frac{1}{8}.
\]
We can then prove the following result.

**Proposition 7.** Assume there are \( N > 2 \) agents facing the risks \( \tilde{x}_1, \ldots, \tilde{x}_N \), all of which exhibit pairwise zero correlation, i.e. \( \text{corr}(\tilde{x}_i, \tilde{x}_j) = 0 \) for all \( i \neq j \). Moreover, assume that after a Pigou-Dalton transfer in risk between agent 1 and 2 the pairwise correlations are still equal to zero: \( \text{corr}(\tilde{x}_a', \tilde{x}_b') = 0 \) for all \( a \neq b \). Then,

- (i) the distribution of fatalities after the Pigou-Dalton transfer may or may not be more catastrophic;
- (ii) the distribution of fatalities after the Pigou-Dalton transfer is more variable.

**Proof.** The counterexample shown above proves (i). To prove (ii), we compute the variance of the distribution of fatalities before and after the Pigou-Dalton transfer for a situation in which all risks have pairwise zero-correlation. We assume \( N = 3 \). Before the transfer, the variance of the distribution of fatalities is given by

\[
\text{var}(\tilde{d}) = \text{var}(\tilde{x}_1) + \text{var}(\tilde{x}_2) + \text{var}(\tilde{x}_3) + 2\rho \sqrt{\text{var}(\tilde{x}_1) \text{var}(\tilde{x}_2)} + 2\text{cov}(\tilde{x}_1, \tilde{x}_3) + 2\text{cov}(\tilde{x}_2, \tilde{x}_3),
\]

(6)

where \( \text{var}(\tilde{x}_i) = p_i(1 - p_i) \) for \( i = \{1, 2, 3\} \). After the transfer, the variance of the distribution of fatalities becomes

\[
\text{var}(\tilde{d}') = \text{var}(\tilde{x}_1') + \text{var}(\tilde{x}_2') + \text{var}(\tilde{x}_3') + 2\rho' \sqrt{\text{var}(\tilde{x}_1') \text{var}(\tilde{x}_2')} + 2\text{cov}(\tilde{x}_1', \tilde{x}_3') + 2\text{cov}(\tilde{x}_2', \tilde{x}_3'),
\]

(7)

with \( \text{var}(\tilde{x}_1') = (p_1 - \delta)(1 - p_1 + \delta) \), \( \text{var}(\tilde{x}_2') = (p_2 + \delta)(1 - p_2 - \delta) \), \( \text{var}(\tilde{x}_3') = p_3(1 - p_3) \).

Using the information on the pairwise zero-correlation, we further simplify expressions (5) and (6) to

\[
\text{var}(\tilde{d}) = p_1(1 - p_1) + p_2(1 - p_2) + \text{var}(\tilde{x}_3),
\]

\[
\text{var}(\tilde{d}') = (p_1 - \delta)(1 - p_1 + \delta) + (p_2 + \delta)(1 - p_2 - \delta) + \text{var}(\tilde{x}_3').
\]

The result underlines that the Pigou-Dalton transfer only affects the variances of \( \tilde{x}_1 \) and \( \tilde{x}_2 \), while \( \text{var}(\tilde{x}_3') = \text{var}(\tilde{x}_3) \) because the distribution \( \tilde{x}_3 \) did not change. It follows that for all \( \delta \in [0, \frac{p_1 - p_2}{2}] \), \( \text{var}(\tilde{d}') > \text{var}(\tilde{d}) \). The generalization of the proof to \( N > 3 \) is straightforward. \( \blacksquare \)

Proposition 7 demonstrates that, for \( N > 2 \), it is impossible to make generic statements about whether the distribution becomes more or less catastrophic based on pairwise correlation coefficients alone. Yet one can conclude on the variability of the distribution of fatalities. In the rest of the paper, we will thus focus our attention on the variability of the distribution of fatalities.
4.4 Pigou-Dalton transfers with correlated risks

In this section, we address situations in which the risks to multiple agents are correlated. As before, we study a situation with \( N > 2 \) agents at risk. A Pigou-Dalton transfer in risk is implemented between agents 1 and 2, with \( p_1 > p_2 \). If there is dependence between the risks in a \( N \)-agent world, one cannot know whether or not the distribution of fatalities becomes more variable after the Pigou-Dalton transfer. The following impossibility result underpins our claim.

**Proposition 8.** Assume there are \( N \) agents facing the risks \( \tilde{x}_1, \ldots, \tilde{x}_N \). The effect of a Pigou-Dalton transfer in risk (from agent 1 to agent 2) on the distribution of fatalities is ambiguous in the following sense: If the risk dependence among agents 3, ..., \( N \) is altered by the transfer, and if \( N \) is large enough, then it is generally impossible to conclude about whether the distribution of fatalities becomes more or less variable.

**Proof.** We study the distribution of fatalities \( \tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_N \). For notational convenience we partition the distribution into \( \tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \tilde{y} \) with \( \tilde{y} := \tilde{x}_3 + \ldots + \tilde{x}_N \). It suffices to show that it is not possible to conclude about whether the distribution becomes more variable when \( N \) is large enough. To do so, we compute the variance before and after the Pigou-Dalton risk transfer. The variance of the pre-transfer distribution of fatalities, \( \text{var}(\tilde{d}) \), and the post-transfer distribution of fatalities, \( \text{var}(\tilde{d}') \), follows from (6) and (7), respectively. We are interested in the change in variance:

\[
\text{var}(\tilde{d}') - \text{var}(\tilde{d}) = \Delta_{PD} + \Delta_{OA} + 2\Delta_{cov},
\]

which we split into three terms.

The first term is the change in variance caused by the Pigou-Dalton transfer:

\[
\Delta_{PD} := \text{var}(\tilde{x}_1' + \tilde{x}_2') - \text{var}(\tilde{x}_1 + \tilde{x}_2);
\]

the second term captures the change in dependence among the other agents not involved in the Pigou-Dalton transfer:

\[
\Delta_{OA} := \text{var}(\tilde{y}') - \text{var}(\tilde{y});
\]

and the third term captures the change in dependence between the two agents involved in the Pigou-Dalton transfer and all the others:

\[
\Delta_{cov} := \text{cov}(\tilde{x}_1' + \tilde{x}_2', \tilde{y}') - \text{cov}(\tilde{x}_1 + \tilde{x}_2, \tilde{y}).
\]

The three terms are not of the same size. If \( N \) is large, the change in variance caused by the Pigou-Dalton transfer \( \Delta_{PD} \) is bounded, whereas \( \Delta_{OA} \) is not. Specifically, for any \( p_1 > p_2 \) and any \( \delta \in [0, \frac{p_1 - p_2}{2}] \),

\[
1 \geq \Delta_{PD} \geq -2.
\]

The proof of (8) is straightforward and thus details are omitted.\(^8\) To compute \( \Delta_{OA} \), we use the two extreme cases identified in Proposition 5, in which \( \tilde{y}' \) is a sum

\[^8\text{Equation (8) follows directly from Lemma 1. There are two possible ways to compute } \Delta_{PD}: \text{ (i) as } \]


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of Bernoulli variables with the respective probabilities \( p_j \) for \( j = 3, \ldots, N \). Let \( \mu_{OA} := p_3 + \ldots + p_N \in [Z, Z + 1[ \) define the expected number of fatalities among the agents not involved in the transfer (with \( Z \) being an integer). Then

\[
m_y \leq \text{var}(\tilde{y'}) \leq M_y,
\]

where the definitions of minimum variance \( m_y := (1 - \mu_{OA} + Z) (\mu_{OA} - Z) \) and maximum variance \( M_y := \left( \sqrt{p_3(1 - p_3)} + \ldots + \sqrt{p_N(1 - p_N)} \right)^2 \) hold for both \( \text{var}(\tilde{y'}) \) and \( \text{var}(\tilde{y}) \), respectively. Both extreme situations are possible in the sense that there exists a change in the dependence structure of the risks faced by agents 3, \ldots, \( N \) such that \( \text{var}(\tilde{y}) \) is either equal to the minimum variance \( m_y \); or equal to the maximum variance \( M_y \). When \( N \to \infty \), the maximum variance \( M_y \) goes to \( +\infty \) and the minimum variance satisfies \( m_y \in [0, 1] \) (because \( 0 \leq \mu_{OA} - Z < 1 \)). Thus,

\[
m_y - M_y \leq \Delta_{OA} \leq M_y - m_y,
\]

where the lower (upper) bound is obtained when \( \tilde{y} \) has maximum (minimum) variance and \( \tilde{y}' \) has minimum (maximum) variance. In other words, the lower bound is equal to the maximum decrease in variance due to the change in the dependence structure among the risks \( \tilde{x}_3, \ldots, \tilde{x}_N \) and the upper bound is the maximum change of variance due to this change in dependence.

The term \( \Delta_{cov} \) may increase variability, but what is more important to notice is that the change in the dependence structure of the risks to the agents not involved in the Pigou-Dalton transfer (\( \Delta_{OA} \)) potentially offsets any other change in variability (\( \Delta_{PD} \) or \( \Delta_{cov} \) or their sum), because the effect is unbounded when \( N \to \infty \).

Proposition 8 states that it is impossible to predict how a Pigou-Dalton transfer in risk alters the degree of variability when the transfer in risk can arbitrarily affect the dependence structure of agents not directly involved in the transfer. In the following, we further constrain the problem and assume that the variance of the sum of the risks of the agents not involved in the transfer is fixed. This additional constraint allows us to formally describe how a Pigou-Dalton transfer between two correlated risks affects the distribution of fatalities. While the equivalence results established in Proposition 9 resemble those established in Proposition 1, it is important to recall that, in general, the notion of “more variable” is less demanding than that of “more catastrophic”.

**Proposition 9.** Assume that the variance of the distribution of fatalities of agents 3, \ldots, \( N \) is not altered by a Pigou-Dalton transfer in risk \( \delta \in [0, \frac{p_2 - p_3}{2}] \) between agent 1 and agent 2. Then, the two following statements are equivalent:

- (i) as the difference between the minimum and maximum variances between the two initial risks \( \tilde{x}_1 \) and \( \tilde{x}_2 \);
- (ii) as the difference between the minimum and maximum variances between the two risks \( \tilde{x}'_1 \) and \( \tilde{x}'_2 \) after the Pigou-Dalton transfer. When \( p_1 + p_2 > 1 \) then the difference between the maximum variance after the transfer and the minimum variance before the transfer is equal to \( 2(\delta + (1 - p_1)) \), which is less than unity under the assumption on the range of \( \delta \). The difference between the maximum variance before the transfer and the minimum variance after the transfer is equal to \( -2(1 - p_1) \), which is larger than \( -2 \). In the case of \( p_1 + p_2 \leq 1 \) the bounds are \( 2(\delta + p_2) \) and \( -2p_2 \) respectively and the same conclusion holds.
(i) the distribution of fatalities is more variable;

(ii) the new correlation \( \rho' \) between \( \tilde{x}'_1 \) and \( \tilde{x}'_2 \) is strictly larger than a critical level of correlation, i.e.

\[
\rho' \geq \rho^* + \frac{\text{cov} \left( \tilde{x}'_1 + \tilde{x}'_2, \sum_{i=3}^{N} \tilde{x}_i \right) - \text{cov} \left( \tilde{x}'_1 + \tilde{x}'_2, \sum_{i=3}^{N} \tilde{x}_i \right)}{\sqrt{(p_1 - \delta)(1 - p_1 + \delta)(p_2 + \delta)(1 - p_2 - \delta)}},
\]

where \( \rho^* \) is the critical level of correlation (1) given in Proposition 3 for the two-agent world.

\textbf{Proof.} By assumption \( \text{var}(\tilde{y}) = \text{var}(\tilde{y}') \). Therefore, \( \text{var}(\tilde{d}') > \text{var}(\tilde{d}) \) iff

\[
\text{var}(\tilde{x}'_1) + \text{var}(\tilde{x}'_2) + 2\rho' \sqrt{\text{var}(\tilde{x}'_1)\text{var}(\tilde{x}'_2)} + 2\text{cov}(\tilde{x}'_1 + \tilde{x}'_2, \tilde{y}) > \text{var}(\tilde{x}_1) + \text{var}(\tilde{x}_2) + 2\rho\sqrt{\text{var}(\tilde{x}_1)\text{var}(\tilde{x}_2)} + 2\text{cov}(\tilde{x}_1 + \tilde{x}_2, \tilde{y}).
\]

Solving for \( \rho' \) yields:

\[
\rho' > \frac{\delta(\delta - p_1 + p_2)}{\sqrt{\text{var}(\tilde{x}'_1)\text{var}(\tilde{x}'_2)}} + \rho \sqrt{\text{var}(\tilde{x}'_1)\text{var}(\tilde{x}'_2)} + \frac{\text{cov}(\tilde{x}_1 + \tilde{x}_2 - \tilde{x}'_1 - \tilde{x}'_2, \tilde{y})}{\sqrt{\text{var}(\tilde{x}'_1)\text{var}(\tilde{x}'_2)}},
\]

where the first two terms on the right hand side are equal to the critical level of correlation \( \rho^* \) found in Proposition 3 for \( N = 2 \).

Some observations on Proposition 9 are warranted. First, note that for the special case analyzed by Keeney (1980) we have \( \rho = \rho' = \text{cov}(\tilde{x}_1 + \tilde{x}_2 - \tilde{x}'_1 - \tilde{x}'_2, \tilde{y}) = 0 \) so that the inequality (9) is satisfied, because \( \delta - p_1 + p_2 < 0 \). More generally, the extension to \( N \) agents does not yield a different result than the one obtained for two agents when

\[
\text{cov} \left( \tilde{x}'_1 + \tilde{x}'_2, \sum_{i=3}^{N} \tilde{x}_i \right) = \text{cov} \left( \tilde{x}'_1 + \tilde{x}'_2, \sum_{i=3}^{N} \tilde{x}_i \right).
\]

In words, Proposition 9 holds whenever the Pigou-Dalton transfer in risk between agent 1 and agent 2 does not affect the correlation between the distribution of fatalities of these two agents and the distribution of fatalities of the other \( N - 2 \) agents. This condition is less restrictive than assuming (as we do in Proposition 6) that the risk dependence of among the agents is fixed. On the other hand, Proposition 9 considers a weaker concept than that of ”more catastrophic”.

5 Conclusion

In this paper we examine the statistical dependence structure of risky social situations. In particular, we explore the relationship between more catastrophic, more correlated, and more equitable risks. To do so, we define a more catastrophic situation
as a mean-preserving spread of the distribution of fatalities, and a more equitable situation as one that results in a smaller difference between the probabilities of death faced by two agents. Our results indicate that a higher correlation between two risks is always equivalent to a more catastrophic situation and a higher probability of simultaneous deaths. We characterize a set of conditions under which risk equity induces a more catastrophic situation. These conditions hold whenever a change in risk equity does not reduce the correlation between the two risks, or when one individual’s fate is certain. This said, our results also pin down the conditions under which a less catastrophic and more equitable situation may be achieved.

The key contribution of our paper is the extension of Keeney (1980)’s result on the ex ante / ex post conflict in risk management to a world with dependent risks. It delivers a simple message: Keeney’s result holds whenever the risk dependence structure is not altered “too much” through the risk transfer; it always holds when the risk dependence structure stays the same. This is an important special case consistent with the traditional comparative statics approach. Indeed, we demonstrate that Keeney’s result is maintained if we compare a risky social situation to a more equitable one while keeping the rest of the world static and, in particular, while keeping risk dependence fixed. We emphasize that the comparative statics approach is problematic in this context as it is conceptually not straightforward to characterize a situation in which risk equity changes, but risk dependence does not. Changing marginal distributions also changes the admissible range of correlation and it is, therefore, fallacious to conceive of risk equity and risk dependence as two separate concepts.

In practice, risk equity and risk dependence are often not orthogonal. Consider the introduction of a new technology—say a better navigation system in cars. This technology will reduce differences in the risk of having a car accident (e.g., by balancing out differences in driving skills or in car safety features). Therefore, the technology will increase risk equity. On the other hand, it may also increase the dependence structure of individual accident risks, e.g. if a software failure affects the navigation system. Hence, the new technology may increase both risk equity and risk interdependence at the same time. It is therefore important to explore what happens if the dependence structure of risky social situations is allowed to vary after a risk transfer. We have seen that general results come at a cost in terms of complexity. Indeed, it is impossible to obtain any result without specifying the change in the risk dependence structure. Characterizing all pairwise correlations is—perhaps unsurprisingly—not enough to derive results on the dependence structure of multiple risks; it is sufficient, however, to study the risk sharing of two agents (in the presence or absence of other agents), or if one is interested in the notion of more variable risks (as opposed to the notion of more catastrophic risks). In this case, Keeney (1980)’s result holds whenever the risk transfer does not affect the correlation between the risk distributions of the two agents involved in the transfer and the distribution of fatalities among the other \(N - 2\) agents.

Whereas we derive our results in the context of mortality risk, the underlying mathematics apply to other managerial decisions that involve risky binary outcomes. One might think of defining the optimal vaccination strategy, or allocating resources to innovation initiatives, or making investment decisions in pharmaceutical labs, to name only a few areas of application. There is mounting experimental evidence that people
care about ex ante as well as ex post tradeoffs in risky social decisions, and that the correlation of individual risks matters (see Rheinberger and Treich 2016). We conclude that the analysis of the dependence structure of social risks is subtle and deserves more attention in future theoretical and empirical studies.
References


Puccetti, G., L. Rüschendorf. 2012. Computation of sharp bounds on the distri-


A Appendix

A.1 Example of independent risks after the risk transfer

We asserted in §3 that it is possible to find examples in which the two risks before the Pigou-Dalton transfer in risk, \( \tilde{x}_1 \) and \( \tilde{x}_2 \), as well as after the transfer, \( \tilde{x}'_1 \) and \( \tilde{x}'_2 \), are uncorrelated and hence independent. Consider the below situations \((X)\) and \((Y)\) with \( S := 16 \) states \((\omega_1, ..., \omega_{16})\).

\[
\begin{bmatrix}
\tilde{x}_1 & \tilde{x}_2 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\tilde{x}'_1 & \tilde{x}'_2 \\
1 & 1 \\
1 & 1 \\
0 & 1 \\
1 & 1 \\
1 & 1 \\
0 & 1 \\
0 & 1 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\((X)\)

\((Y)\)

The individual risks \( \tilde{x}_1 \) and \( \tilde{x}_2 \) in the initial situation \((X)\) are independent, and so are the individual risks \( \tilde{x}'_1 \) and \( \tilde{x}'_2 \) after a Pigou-Dalton transfer of \( \delta = \frac{1}{4} \). When the risks are independent before and after the Pigou-Dalton transfer, as in situations \((X)\) and \((Y)\), the distribution of fatalities \( \tilde{d}' = \tilde{x}'_1 + \tilde{x}'_2 \) is more catastrophic than \( \tilde{d} = \tilde{x}_1 + \tilde{x}_2 \) as predicted by Keeney (Proposition 2).

\[^9\text{In settings with more than two risks, zero correlation does not necessarily imply independence (see situations \((J)\) and \((K)\) in \(i\sigma e4.3\). Yet for two Bernoulli random variables zero correlation always implies independence.}\]
A.2 Proof of statement (i) in Proposition 4 when $\rho < 0$

From (1) we have that

$$\rho^*(\delta) := \frac{\delta(p_2 - p_1 + \delta) + \rho \sqrt{p_1 \sqrt{1 - p_1} \sqrt{p_2 \sqrt{1 - p_2}}}}{\sqrt{1 - p_1} + \delta \sqrt{p_1 - \delta \sqrt{p_2 + \delta \sqrt{1 - p_2}}}}.$$

Using Proposition 3, we need to show that $\rho' \geq \rho^*(\delta)$ for all $\delta \in (0, \frac{p_1 - p_2}{2})$. As we assume that $\rho' \geq \rho$, it is enough to show that $\rho \geq \rho^*(\delta)$. Note that $\rho^*(0) = \rho$, thus we study the sensitivity of $\rho^*(\delta)$ to $\delta$. We first compute the derivative of $\rho^*(\delta)$ with respect to $\delta$:

$$\frac{\partial \rho^*(\delta)}{\partial \delta} = \frac{(p_1 - p_2 - 2\delta)(A(\delta)\rho + B(\delta))}{2[(p_1 - \delta)(1 - p_1 + \delta)(1 - p_2 - \delta)(p_2 + \delta)]^{\frac{3}{2}}} \quad (10)$$

where

$$A(\delta) := \sqrt{p_1(1 - p_1)p_2(1 - p_2)(2\delta^2 + 2\delta(p_2 - p_1) - (1 + 2p_1p_2 - p_2 - p_1))}$$

and

$$B(\delta) := (p_1p_2 + (1 - p_1)(1 - p_2))(\delta^2 + \delta(p_2 - p_1) - 2p_1p_2) + 2p_1^2p_2^2.$$

Observe then that the sign of $\delta \mapsto \frac{\partial \rho^*(\delta)}{\partial \delta}$ over $[0, \frac{p_1 - p_2}{2}]$ is the same as the sign of the function $\delta \mapsto A(\delta)\rho + B(\delta)$ for $\delta \in (0, \frac{p_1 - p_2}{2})$.

**Lemma 2.** When $\rho < 0$, the function $\delta \mapsto A(\delta)\rho + B(\delta)$ satisfies the following property

$$\forall \delta \in (0, \frac{p_1 - p_2}{2}), \quad A(\delta)\rho + B(\delta) < 0. \quad (11)$$

Proof of Lemma 2. From the expressions of $A(\delta)$ and $B(\delta)$, we know that $A(\delta)\rho + B(\delta)$ is a second-degree polynomial of $\delta$. Differentiating $\delta \mapsto A(\delta)\rho + B(\delta)$ gives an affine equation in $\delta$ and solving for its zero gives $\delta = \frac{p_1 - p_2}{2}$. The function $A(\delta)\rho + B(\delta)$ achieves its minimum at this value. In addition, we now prove that

$$A(0)\rho + B(0) < 0 \quad A'(0)\rho + B'(0) < 0. \quad (12)$$

Thus (11) follows because of the properties of a polynomial of the second degree in $\delta$ (it is decreasing over the interval $[0, \frac{p_1 - p_2}{2}]$ and thus takes only negative values over this interval).

To prove (12), we need the expressions of $A(0)$, $B(0)$, $A'(0)$ and $B'(0)$:

$$A(0) = -\sqrt{p_1(1 - p_1)p_2(1 - p_2)(p_1p_2 + (1 - p_1)(1 - p_2))}$$
$$B(0) = -2p_1p_2(1 - p_1)(1 - p_2)$$
$$A'(0) = 2(p_2 - p_1)\sqrt{p_1(1 - p_1)p_2(1 - p_2)}$$
$$B'(0) = (p_2 - p_1)(p_1p_2 + (1 - p_1)(1 - p_2)).$$

We distinguish two cases:
Case 1: when \( p_1 + p_2 \leq 1 \). Then from Lemma 1, \( \rho \geq \frac{-\sqrt{p_1 p_2}}{\sqrt{1 - p_1} \sqrt{1 - p_2}} \) then

\[
A(0) + \rho B(0) \leq p_1 p_2 (p_1 + p_2 - 1) \leq 0
\]

\[
A'(0) + \rho B'(0) \leq (p_2 - p_1)(1 - p_1 - p_2) < 0
\]

because \( p_2 < p_1 \).

Case 2: when \( p_1 + p_2 > 1 \). Then from Lemma 1, \( \rho \geq \frac{-\sqrt{1 - p_1} \sqrt{1 - p_2}}{p_1 p_2} \) then

\[
A(0) + \rho B(0) \leq (1 - p_1)(1 - p_2)(1 - p_1 + p_2) < 0
\]

\[
A'(0) + \rho B'(0) \leq (p_2 - p_1)(p_1 + p_2 - 1) < 0
\]

because \( p_2 < p_1 \).

Proof of statement (i) in Proposition 4 when \( \rho < 0 \). Since \( \delta \leq \frac{p_1 - p_2}{2} \), then from the expression (10) and Lemma 2, it is clear that

\[
\forall \delta \in \left[0, \frac{p_1 - p_2}{2}\right] \quad \frac{\partial \rho^*(\delta)}{\partial \delta} \leq 0. \tag{13}
\]

When \( \delta = 0 \), then \( \rho^* = \rho \). Using the fact that \( \rho^* \) is decreasing in \( \delta \) (i.e. (13)), then for all \( \delta \in \left[0, \frac{p_1 - p_2}{2}\right] \), \( \rho^*(\delta) \leq \rho^*(0) = \rho \leq \rho' \). Since \( \rho' \) satisfies (1), then the distribution of fatalities is more catastrophic and (i) is proved.

\[\blacksquare\]

A.3 Proof of statement (ii) in Proposition 5

The proof of Proposition 5 is inspired from Lemma 3.1 in Bernard et al. (2016). We use here the fact that the resulting distribution is less catastrophic and not only less variable as it is the case in their paper.

Lemma 3 (Least catastrophic distribution of fatalities). Define for \( j = 1, \ldots, N \),

\[
a_j = \left( \sum_{i=1}^{j} p_i \right) \mod 1,
\]

and the sets

\[
I_j = \begin{cases} [a_{j-1}, a_j] & \text{if } a_j > a_{j-1} \\ [0, a_j] \cup [a_{j-1}, 1] & \text{if } a_j < a_{j-1} \end{cases},
\]

where we define \( a_0 = 0 \). Then, the least catastrophic distribution is \( \tilde{\mathbf{d}}^a := \sum_{j=1}^{N} \tilde{y}_j \) where \( \tilde{y}_j \) are defined by

\[
\tilde{y}_j = \mathbb{1}_{\tilde{u} \in I_j}, \tag{14}
\]

where \( \tilde{u} \) is a standard uniformly distributed random variable over \((0,1)\). Furthermore, \( \tilde{d}^a \) takes only two values \( M \) with probability \( p_M = M + 1 - \mu \) and \( M + 1 \) with probability \( 1 - p_M \) where \( M = \lfloor \mu \rfloor \) (largest integer inferior or equal to \( \mu \)).
Proof. Let us first observe that $\tilde{y}_j$ defined by (14) are Bernoulli with parameter $p_j$. Furthermore, $\tilde{d}^a = \tilde{y}_1 + \tilde{y}_2 + \cdots + \tilde{y}_N$ only takes values $M$ with probability $p_M$ or $(M + 1)$ with probability $1 - p_M$ (where $p_M = 1$ may hold if it is constant). Consider any other distribution of fatalities $\tilde{d} = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_N$ with $\tilde{x}_j$ being a Bernoulli distribution with parameter $p_j$ and let us show that $\tilde{d}$ is more catastrophic than $\tilde{d}^a$. Observe that any such distribution of fatalities $\tilde{d}$ takes values in $\{0, 1, 2, \ldots, N\}$.

It is clear that $\forall x \in ]0, M[\), $F_{\tilde{d}}(x) \geq F_{\tilde{d}^a}(x) = 0$ and $\forall x \in [(M + 1), +\infty[\), $F_{\tilde{d}}(x) \leq F_{\tilde{d}^a}(x) = 1$. Since $F_{\tilde{d}}(x)$ and $F_{\tilde{d}^a}(x)$ are constant on the interval $[M, M + 1]$ one has

$$\exists c \geq 0, \begin{cases} \forall x \in (0, c), & F_{\tilde{d}}(x) \geq F_{\tilde{d}^a}(x) \\ \forall x \in (c, +\infty), & F_{\tilde{d}}(x) \leq F_{\tilde{d}^a}(x) \end{cases}$$

(15)

namely, $c = M + 1$ if $F_{\tilde{d}}(M) > F_{\tilde{d}^a}(x)$ and $c = M$ if $F_{\tilde{d}}(M) \leq F_{\tilde{d}^a}(x)$. In other words, the distribution function $F_{\tilde{d}}$ crosses $F_{\tilde{d}^a}$ exactly once from above. Since $E[\tilde{d}] = E[\tilde{d}^a]$ this implies the well-known one-crossing property that characterizes second-order stochastic dominance. \(\blacksquare\)