Extreme Value Analysis of the Haezendonck–Goovaerts Risk Measure with a General Young Function

Qihe Tang\textsuperscript{[a]} and Fan Yang\textsuperscript{[b]} *

\textsuperscript{[a]} Department of Statistics and Actuarial Science, University of Iowa
241 Schaeffer Hall, Iowa City, IA 52242, USA
\textsuperscript{[b]} Actuarial Science Program
College of Business and Public Administration, Drake University
345 Aliber Hall, 2507 University Avenue, Des Moines, IA 50311, USA

March 10, 2014

Abstract

For a risk variable $X$ and a normalized Young function $\varphi(\cdot)$, the Haezendonck–Goovaerts risk measure for $X$ at level $q \in (0, 1)$ is defined as

$$H_q[X] = \inf_{x \in \mathbb{R}} (x + h),$$

where $h$ solves the equation $E[\varphi((X - x)_+ / h)] = 1 - q$ if $\Pr(X > x) > 0$ or is 0 otherwise. In a recent work, we implemented an asymptotic analysis for $H_q[X]$ with a power Young function for the Fréchet, Weibull and Gumbel cases. A key point of the implementation is that $h$ can be explicitly solved for fixed $x$ and $q$, which gives rise to the possibility to express $H_q[X]$ in terms of $x$ and $q$. For a general Young function, however, this approach does not work any more and the problem becomes a lot harder. In the present paper, we extend the asymptotic analysis for $H_q[X]$ to the case with a general Young function and we establish a unified approach for the three extreme value cases. In doing so, we overcome several technical difficulties mainly due to the intricate relationship between the working variables $x$, $h$ and $q$.

Keywords: asymptotics; Haezendonck–Goovaerts risk measure; max-domain of attraction; (extended) regular variation; Young function

1 Introduction

Let $X$ be a real-valued random variable, representing a risk variable in loss–profit style, with a distribution function $F = 1 - F$ on $\mathbb{R} = (-\infty, \infty)$. Let $\varphi(\cdot)$ be a normalized Young function; that is $\varphi(\cdot)$ a nonnegative and convex function on $\mathbb{R}_+ = [0, \infty)$ with $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(\infty) = \infty$. Due to its convexity, the Young function $\varphi(\cdot)$ is continuous and strictly increasing on $\{t \in \mathbb{R}_+ : \varphi(t) > 0\}$. See Krasnosel’skiĭ and Rutickiĭ (1961) and Neveu

*Corresponding author: Fan Yang; E-mail: fan.yang@drake.edu; Cell: 319-471-0811; Fax: 515-271-4518
(1975) for related discussions of Young functions. Recall that the Orlicz heart associated with the Young function \( \varphi(\cdot) \) is defined as

\[
L^\varphi_0 = \{ X : \mathbb{E}[\varphi(cX)] < \infty \text{ for all } c > 0 \};
\]

see, e.g. page 77 of Rao and Ren (1991). By the convexity of \( \varphi(\cdot) \) we know that the Orlicz heart \( L^\varphi_0 \) is a convex set. Thus, for \( X \in L^\varphi_0 \), the expectation \( \mathbb{E}[\varphi((X-x)_+/h)] \), which frequently appears in the sequel, is finite for every \( x \in \mathbb{R} \) and \( h > 0 \). Here, and throughout the paper, for a real number \( x \) we write \( x_+ = x 1_{\{x \geq 0\}} \) as its positive part, with \( 1_A \) denoting the indicator of an event \( A \).

For a Young function \( \varphi(\cdot) \) and a risk variable \( X \in L^\varphi_0 \), let \( h = h(x,q) \) be the unique solution to the equation

\[
\mathbb{E}\left[ \varphi\left(\frac{(X-x)_+}{h}\right)\right] = 1 - q, \quad q \in (0,1),
\]

if \( F(x) > 0 \) and let \( h(x,q) = 0 \) if \( F(x) = 0 \). For \( q \in (0,1) \), the Haezendonck–Goovaerts (HG) risk measure for \( X \) is defined as

\[
H_q[X] = \inf_{x \in \mathbb{R}} (x + h(x,q)).
\]

This risk measure was first introduced by Haezendonck and Goovaerts (1982) based on the Swiss premium calculation principle induced by the Orlicz norm and was revisited by Goovaerts et al. (2004). Since recently, it has attracted increasing attention from researchers; see Bellini and Rosazza Gianin (2008a, 2008b, 2012), Krätschmer and Zähle (2011), Nam et al. (2011), Goovaerts et al. (2012), Tang and Yang (2012), Mao and Hu (2012), and Ahn and Shyamalkumar (2014), among others.

For a convex Young function \( \varphi(\cdot) \), the HG risk measure is a law invariant and coherent risk measure. Furthermore, if the Young function \( \varphi(\cdot) \) is strictly convex, then the minimizer \( (x_*,h_*) \) in (1.2) is unique. See Proposition 3 (d) and (e) of Bellini and Rosazza Gianin (2012) for these assertions. The simplest case of the HG risk measure is when \( \varphi(s) = s \lor 0 \), reducing to the well-known Tail Value-at-Risk.

In a recent work, we implemented an asymptotic analysis for \( H_q[X] \) with a power Young function for the Fréchet, Weibull and Gumbel cases. The assumption of a power Young function essentially simplifies the question. Precisely speaking, through (1.1) the variable \( h \) can be expressed in terms of \( x \) and \( q \). Plugging this formula into (1.2), the computation of the HG risk measure reduces to a one-variable minimization problem.

The aim of this paper is to extend this asymptotic study of the HG risk measure to a general Young function. This extension brings a lot of technical difficulties. For this general case, an explicit formula for \( h \) in terms of \( x \) and \( q \) cannot be deduced from (1.1) any more. Without such a formula, the approach of Tang and Yang (2012) fails.
Our main methodology is based on extreme value theory. We derive asymptotics for the HG risk measure at a high confidence level for the risk variable \( X \) following a distribution function from the max-domain of attraction of the generalized extreme value distribution. Different from Tang and Yang (2012), we chase a unified treatment for the Fréchet, Gumbel and Weibull cases.

Denote by \( \hat{x} \leq \infty \) the upper endpoint of the risk variable \( X \). It is easy to see that if \( \Pr (X = \hat{x}) > 0 \), then \( H_q[X] = \hat{x} \) as \( q \) is close enough to 1. Therefore we will only consider the non-trivial case that \( \Pr (X = \hat{x}) = 0 \). Let \((x_*, h_*)\) be the unique minimizer for the question of (1.2) subject to (1.1). In Bellini and Rosazza Gianin (2012), the minimizer \( x_* \) is called the Orlicz quantile of \( X \). Hence, the HG risk measure for \( X \) is given by \( x_* + h_* \).

For the power Young function case, the equivalence between the limits \( x_* \rightarrow \hat{x} \) and \( q \uparrow 1 \) easily follows from Lemma 2.2 of Tang and Yang (2012) and it provides a lot of convenience in the derivations there. In the current general Young function case, we shall prove that \( x_* \rightarrow \hat{x} \) still holds as \( q \uparrow 1 \), which is an important preparation for the later derivations. A key step of the current work is that, for a distribution function \( F \) from the max-domain of attraction of a generalized extreme value distribution with an auxiliary function \( a(\cdot) \), the ratio of \( a(1/F(x_*)) \) and \( h_* \) is proved to converge to a positive constant as \( q \uparrow 1 \). This constant ratio enables us to get rid of \( h_* \) in our derivations.

The rest of the paper consists of six sections. Section 2 prepares some preliminaries on regular variation, extended regular variation, and the max-domain of attraction of the generalized extreme value distribution. Section 3 shows two main results, which represent a unified asymptotic solution of the HG risk measure with a general Young function for the Fréchet, Gumbel and Weibull cases. Section 4 lists some lemmas that are used to complete the proof of the first main result. Sections 5-6 focus on the relationships between the three working variables \( x, h \) and \( q \) through the auxiliary function \( a(\cdot) \), which are key steps in the proof of the second main result. Finally, Section 7 numerically examines the accuracy of the asymptotic formula obtained in the paper.

2 Preliminaries

2.1 Notational conventions

For two positive functions \( f(\cdot) \) and \( g(\cdot) \), we write \( f(\cdot) \sim g(\cdot) \) if the ratio of the left-hand side (LHS) and right-hand side (RHS) converges to 1, that is, \( \lim f(\cdot)/g(\cdot) = 1 \). We also write \( f(\cdot) \lesssim g(\cdot) \) if \( \limsup f(\cdot)/g(\cdot) \leq 1 \) and write \( f(\cdot) \asymp g(\cdot) \) if \( 0 < \liminf f(\cdot)/g(\cdot) \leq \limsup f(\cdot)/g(\cdot) < \infty \).

By saying that \( \varepsilon \) is a small positive number we mean that \( \varepsilon \) is an arbitrarily small but fixed positive number. For example, with this convention, if \( \gamma < \alpha^{-1} \wedge \beta^{-1} \), as assumed in Theorem 3.2, then the inequality \( \gamma + \varepsilon < \alpha^{-1} \wedge \beta^{-1} \) will be used without explanation.
We often use the letter $C$ to denote an absolute positive constant, which does not depend on the working variables such as $x$, $h$ and $q$, but which can differ from place to place so that relations such as $C + 1 = C$, $2C = C$ and $C/C = C$ make sense.

The càglàd inverse function of a non-decreasing function $f$ on $\mathbb{R}$ is denoted by $f^{-}(y) = \inf\{x \in \mathbb{R} : f(x) \geq y\}$ for $y \in \mathbb{R}$, where we follow the convention that $\inf \emptyset = +\infty$.

### 2.2 Regular variation

A positive measurable function $f(\cdot)$ is said to be regularly varying at $t_0 = 0+$ or $\infty$ with index $\alpha \in \mathbb{R}$, denoted by $f(\cdot) \in \text{RV}_\alpha(t_0)$, if

$$\lim_{t \to t_0} \frac{f(st)}{f(t)} = s^\alpha, \quad s > 0.$$ 

The class $\text{RV}_0(t_0)$ consists of functions slowly varying at $t_0$. Moreover, a positive measurable function $f(\cdot)$ is said to be rapidly varying at $t_0 = 0+$ or $\infty$, denoted by $f(\cdot) \in \text{RV}_\infty(t_0)$ or $1/f(\cdot) \in \text{RV}_{-\infty}(t_0)$, if

$$\lim_{t \to t_0} \frac{f(st)}{f(t)} = \infty, \quad s > 1.$$

The following Potter’s bounds for regularly varying functions are well known; see Theorem 1.5.6 of Bingham et al. (1987):

**Lemma 2.1** Let $f(\cdot) \in \text{RV}_\alpha(t_0)$ with $t_0 = 0+$ or $\infty$ and $\alpha \in \mathbb{R}$. It holds for arbitrarily small $0 < \varepsilon < 1$ and all $s$, $t$ sufficiently close to $t_0$ that

$$(1 - \varepsilon) \left( \left( \frac{t}{s} \right)^{\alpha + \varepsilon} \wedge \left( \frac{t}{s} \right)^{\alpha - \varepsilon} \right) \leq \frac{f(t)}{f(s)} \leq (1 + \varepsilon) \left( \left( \frac{t}{s} \right)^{\alpha + \varepsilon} \vee \left( \frac{t}{s} \right)^{\alpha - \varepsilon} \right).$$

Fixing $s$ to some constant in the above yields that, for some large constant $C > 0$ and all $t$ sufficiently close to $t_0$,

$$\frac{1}{C} \left( t^{\alpha + \varepsilon} \wedge t^{\alpha - \varepsilon} \right) \leq f(t) \leq C \left( t^{\alpha + \varepsilon} \vee t^{\alpha - \varepsilon} \right). \quad (2.1)$$

The following lemma is a restatement of Proposition 0.8(V) of Resnick (1987):

**Lemma 2.2** Let $f(\cdot)$ be a non-decreasing function on $\mathbb{R}_+$ with $f(\infty) = \infty$. Then $f(\cdot) \in \text{RV}_\alpha(\infty)$ with $\alpha \in \mathbb{R}_+$ if and only if $f^{-}(\cdot) \in \text{RV}_{1/\alpha}(\infty)$, where we follow the convention $1/0 = \infty$ and $1/\infty = 0$. 
2.3 Extended regular variation

The concept of extended regular variation is useful for our unified derivation. By definition, a positive measurable function \( f(\cdot) \) is said to be extended regularly varying at \( \infty \) with index \( \gamma \in \mathbb{R} \), denoted by \( f(\cdot) \in \text{ERV}_\gamma \), if there exists an auxiliary function \( a(\cdot) > 0 \) such that, for all \( s > 0 \),

\[
\lim_{t \to \infty} \frac{f(st) - f(t)}{a(t)} = \frac{s^\gamma - 1}{\gamma},
\]

where the RHS is interpreted as \( \log s \) when \( \gamma = 0 \). See Chapter 3 of Embrechts et al. (1997) or Appendix B of de Haan and Ferreira (2006) for this concept. The auxiliary function \( a(\cdot) \) is often chosen to be

\[
a(t) = \begin{cases} 
\gamma f(t), & \gamma > 0, \\
 f(t) - t^{-1} \int_0^t f(u)du, & \gamma = 0, \\
-\gamma(f(\infty) - f(t)), & \gamma < 0.
\end{cases}
\]

Note that, for \( \gamma = 0 \), as \( t \to \infty \), we have \( a(t) = o(f(t)) \) provided \( f(\infty) = \infty \), while \( a(t) = o(f(\infty) - f(t)) \) provided \( f(\infty) < \infty \).

The following lemma is a restatement of Theorem B.2.18 of de Haan and Ferreira (2006), originally attributed to Drees (1998):

**Lemma 2.3** Let \( f(\cdot) \in \text{ERV}_\gamma \) for \( \gamma \in \mathbb{R} \), namely, relation (2.2) holds for an auxiliary function \( a(\cdot) \) given in (2.3) and for all \( s > 0 \). It holds for every small \( \varepsilon, \delta > 0 \), some \( t_0 = t_0(\varepsilon, \delta) > 0 \) and all \( s, t \) with \( t > t_0 \), \( st > t_0 \) that

\[
\left| \frac{f(st) - f(t)}{a(t)} - \frac{s^\gamma - 1}{\gamma} \right| \leq \varepsilon \left( s^{\gamma+\delta} \lor s^{\gamma-\delta} \right).
\]

Taking supremum of both sides of inequality (2.4) with respect to \( s \) over a closed positive interval containing 1, and then letting the interval boil down to 1, we can easily prove that

\[
\lim_{t \to \infty} \frac{f(t \pm 0) - f(t)}{a(t)} = 0.
\]

It follows from (2.2) and (2.5) that, for all \( s > 0 \),

\[
\lim_{t \to \infty} \frac{f(st) - f(t \pm 0)}{a(t)} = \frac{s^\gamma - 1}{\gamma}.
\]

2.4 Max-domains of attraction

A distribution function \( F \) is said to belong to the max-domain of attraction of a non-degenerate distribution function \( G \), denoted by \( F \in \text{MDA}(G) \), if for a simple sample of size \( n \) from \( F \), its normalized maximum has a distribution weakly converging to \( G \). The classical Fisher–Tippett theorem, attributed to Fisher and Tippett (1928) and Gnedenko
(1943), states that $G$ has to be the generalized extreme value distribution whose standard structure is given by

$$G_\gamma(t) = \exp \left\{ - (1 + \gamma t)^{-1/\gamma} \right\}, \quad \gamma \in \mathbb{R}, 1 + \gamma t > 0,$$

where the RHS is interpreted as $\exp \{ -e^{-t} \}$ when $\gamma = 0$. The generalized extreme value distribution corresponds to the Fréchet, Gumbel and Weibull cases, when $\gamma > 0, \gamma = 0$ and $\gamma < 0$, respectively.

It is well known that $F \in \text{MDA}(G_\gamma)$ with $\gamma > 0$ if and only if $F(\cdot) \in \text{RV}^{-1/\gamma}(\infty)$, while $F \in \text{MDA}(G_\gamma)$ with $\gamma < 0$ if and only if $\hat{x} < \infty$ and $F(\hat{x} - \cdot) \in \text{RV}_{1/\gamma}(0+)$. Moreover, for $F \in \text{MDA}(G_0)$, we have $F(\cdot) \in \text{RV}_-\infty(\infty)$ provided $\hat{x} = \infty$ or $F(\hat{x} - \cdot) \in \text{RV}_\infty(0+)$ provided $\hat{x} < \infty$. Hence, if $\gamma \leq 0$ then $E[X_+^p] < \infty$ for all $p > 0$, while if $\gamma > 0$ then $E[X_+^p] < \infty$ for all $0 < p < 1/\gamma$. See, for example, Chapter 3 of Embrechts et al. (1997) for these statements.

The MDA of the generalized extreme value distribution is related to extended regular variation through

$$U(t) = \left( \frac{1}{F} \right)^\leftarrow (t) = F^\leftarrow \left( 1 - \frac{1}{t} \right), \quad t > 1.$$

We have $F \in \text{MDA}(G_\gamma)$ if and only if $U(\cdot) \in \text{ERV}_\gamma$ with the auxiliary function $a(\cdot)$ given in (2.3) in terms of $U(\cdot)$; see Theorem 1.1.6 of de Haan and Ferreira (2006) for this result.

### 3 Main Results

Recall that $h = h(x, q)$ is the unique solution to equation (1.1) if $F(x) > 0$. In what follows, we often write $h(x) = h(x, q)$ if $q$ is fixed or write $h(q) = h(x, q)$ if $x$ is fixed, as long as doing so causes no confusion. As usual, we say that $\varphi(\cdot)$ is differentiable over $\mathbb{R}_+$ if $\varphi(\cdot)$ is differentiable over $(0, \infty)$ and $\varphi'(0) > 0$ exists.

**Theorem 3.1** Let the Young function $\varphi(\cdot)$ be strictly convex and continuously differentiable over $\mathbb{R}_+$ with $\varphi'(0) = 0$, and let the random variable $X \in L_0^\varphi$. Then the HG risk measure is equal to

$$H_q[X] = x_* + h_*,$$  \hspace{1cm} (3.1)

where the pair $(x_*, h_*)$ solves (1.1), that is,

$$E \left[ \varphi \left( \frac{(X - x)_+}{h} \right) \right] = 1 - q,$$  \hspace{1cm} (1.1)

and

$$E \left[ \varphi' \left( \frac{(X - x)_+}{h} \right) \right] = E \left[ \varphi' \left( \frac{(X - x)_+}{h} \right) \frac{(X - x)_+}{h} \right].$$  \hspace{1cm} (3.2)
**Proof.** By Lemma 4.3, \( h(x) \) is continuously differentiable over \((-\infty, \hat{x})\). Thus, for the minimizer \((x, h)\) of the problem (1.2) subject to (1.1), we have

\[
h'(x) = -1.
\]  

(3.3)

Applying (4.1), (4.2) and (3.3), the derivative of the LHS of (1.1) with respect to \( x \) is equal to

\[
-\frac{1}{h} E \left[ \varphi' \left( \frac{X - x}{h} \right) 1_{(X>x)} \right] + E \left[ \varphi' \left( \frac{X - x}{h} \right) \frac{X - x}{h^2} 1_{(X>x)} \right].
\]

Since the RHS of (1.1) is \( 1 - q \), independent of \( x \), letting the above be 0 yields (3.2) since \( \varphi'_+(0) = 0 \).

Theorem 3.1 is the starting point of our derivations in the current work. Slightly different versions of this result can be found in Remark 7 of Bellini and Rosazza Gianin (2012) and the proof of Corollary 1 of Ahn and Shyamalkumar (2014). By the way, the finiteness of the expectations appearing in Theorem 3.1 and its proof is implied by the proofs of Lemmas 4.1 and 4.2.

Now we restrict our attention to \( F \in \text{MDA}(G_\gamma) \) for \( \gamma \in \mathbb{R} \). By Theorem 1.1.6 of de Haan and Ferreira (2006), \( F \in \text{MDA}(G_\gamma) \) if and only if

\[
\lim_{x \uparrow \hat{x}} \frac{F(x + ya(1/F(x)))}{F(x)} = (1 + \gamma y)^{-1/\gamma} \quad \text{for all } y \text{ with } 1 + \gamma y > 0,
\]  

(3.4)

where the RHS is understood as \( e^{-y} \) when \( \gamma = 0 \). Define a positive random variable \( Y \) distributed

\[
\Pr(Y \leq y) = 1 - (1 + \gamma y)^{-1/\gamma} \quad \text{for all } y > 0 \text{ and } 1 + \gamma y > 0.
\]

Relation (3.4) implies that

\[
\frac{X - x}{a(1/F(x))} \bigg|_{(X > x)} \xrightarrow{d} Y, \quad x \uparrow \hat{x}.
\]

(3.5)

In addition to the conditions of Theorem 5.1, further assume that \( F \in \text{MDA}(G_\gamma) \) with \(-\infty < \gamma < \alpha^{-1} \wedge \beta^{-1}\). By Lemma 6.2, the equation

\[
E[\varphi'(\lambda Y)] = E[\varphi'(\lambda Y) \lambda Y], \quad \lambda > 0,
\]  

(3.6)

has a unique positive solution.

Next we present the main result of this paper.

**Theorem 3.2** In addition to the conditions of Theorem 3.1, assume that \( \varphi(\cdot) \in \text{RV}_\alpha(0+) \cap \text{RV}_\beta(\infty) \) for some \( 1 < \alpha, \beta < \infty \) and \( F \in \text{MDA}(G_\gamma) \) with \(-\infty < \gamma < \alpha^{-1} \wedge \beta^{-1}\). In case
\( \gamma \leq -1 \), further assume that \( \varphi(\cdot) \) is twice differentiable over \( \mathbb{R}_+ \) such that \( E[\varphi''(\lambda Y)] < \infty \) for all \( \lambda > 0 \). Then the pair \((x^*, h^*)\) appearing in (3.1) satisfies

\[
\overline{F}(x^*) \sim \frac{1 - q}{E[\varphi(\lambda Y)]} \quad \text{and} \quad h^* \sim \frac{a(1/F(x^*))}{\lambda}, \quad q \uparrow 1,
\]

where \( \lambda \) is the unique positive solution of (3.6) and \( a(\cdot) \) is given by (2.3) with \( f \) replaced by \( U(\cdot) = \frac{1}{F(x^*)} \).

**Proof.** The second relation for \( h^* \) is proved by Theorem 6.1. Now we prove the first relation. By relation (3.5) and Theorem 6.1, we have

\[
1 - q = E\left[ \varphi\left(\frac{(X - x^*)_+}{h^*_+} \right) \right] = \overline{F}(x^*)E\left[ \varphi\left(\frac{X - x^*}{a(1/F(x^*))} \right) \right| X > x^*] \sim \overline{F}(x^*)E[\varphi(\lambda Y)].
\]

This gives the first relation for \( \overline{F}(x^*) \). \( \square \)

**Corollary 3.1** Under the conditions of Theorem 3.2, as \( q \uparrow 1 \),

(i) (the Fréchet case) for \( \gamma > 0 \),

\[
H_q[X] \sim \left(1 + \frac{\gamma}{\lambda}\right) \left(\int_0^\infty \left(1 + \frac{\gamma}{\lambda}z\right)^{-1/\gamma} d\varphi(z)\right)^\gamma F^{\leftarrow}(q);
\]

(ii) (the Gumbel case) for \( \gamma = 0 \), if \( \hat{x} = \infty \) then

\[
H_q[X] \sim F^{\leftarrow}\left(1 - \frac{1 - q}{\int_0^\infty e^{-z/\lambda} d\varphi(z)}\right);
\]

while if \( \hat{x} < \infty \) then

\[
\hat{x} - H_q[X] \sim \hat{x} - F^{\leftarrow}\left(1 - \frac{1 - q}{\int_0^\infty e^{-z/\lambda} d\varphi(z)}\right);
\]

(iii) (the Weibull case) for \( \gamma < 0 \),

\[
\hat{x} - H_q[X] \sim \left(1 + \frac{\gamma}{\lambda}\right) \left(\int_0^{-\lambda/\gamma} \left(1 + \frac{\gamma}{\lambda}z\right)^{-1/\gamma} d\varphi(z)\right)^\gamma (\hat{x} - F^{\leftarrow}(q)).
\]

**Proof.** By Theorem 3.2, it remains to identify the quantities \( E[\varphi(\lambda Y)] \) and \( a(1/F(x^*)) \) for the three cases, respectively.

(i) Consider the case \( \gamma > 0 \).

\[
E[\varphi(\lambda Y)] = -\int_0^\infty \varphi(\lambda y) d(1 + \gamma y)^{-1/\gamma} = \int_0^\infty \left(1 + \frac{\gamma}{\lambda}z\right)^{-1/\gamma} d\varphi(z).
\]
By (2.3), we have \( a(t) = \gamma U(t) \). Notice that \( U(t) = (F)^{-} (F(x)) \sim x \) by Lemma 3.1 of Tang and Yang (2012). Thus,

\[
H_{q}[X] \sim \left( 1 + \frac{\gamma}{\lambda} \right)x \sim \left( 1 + \frac{\gamma}{\lambda} \right) \left( \int_{0}^{\infty} \left( 1 + \frac{\gamma}{\lambda}z \right)^{-1/\gamma} \, d\varphi(z) \right)^{\gamma} F^{\rightarrow}(q),
\]

where the second equivalent relation is due to \((F)^{-} \in RV_{-\gamma}(+0)\) by Lemma 2.2.

(ii) Consider the case \( \gamma = 0 \).

\[
E[\varphi(\lambda Y)] = -\int_{0}^{\infty} \varphi(\lambda y) \, de^{-y} = \int_{0}^{\infty} e^{-z/\lambda} \, d\varphi(z).
\]

For \( \hat{x} = \infty \), since \( a(t) = o(U(t)) \) and \( U(t) \sim x \), we have \( H_{q}[X] \sim x \). For \( \hat{x} < \infty \), since \( a(t) = o(U(\infty) - U(t)) \), \( U(\infty) = \hat{x} \) and \( U(\infty) - U(t) \sim \hat{x} - x \), we have \( \hat{x} - H_{q}[X] \sim \hat{x} - x \).

(iii) Consider the case \( \gamma < 0 \).

\[
E[\varphi(\lambda Y)] = -\int_{0}^{\infty} \varphi(\lambda y) \, d(1 + \gamma y)^{-1/\gamma} = \int_{0}^{\infty} \left( 1 + \frac{\gamma}{\lambda}z \right)^{-1/\gamma} \, d\varphi(z).
\]

We have \( a(t) = -\gamma (U(\infty) - U(t)) \sim -\gamma (\hat{x} - x) \).

4 Lemmas for Proving Theorem 3.1

In this section we establish some lemmas that are not only used in the proof of Theorem 3.1 but also interesting in their own right. The following two elementary results concern differentiation under the integral sign. The first lemma can easily be retrieved from the proofs of Proposition 1(c) and Corollary 3 of Bellini et al. (2014). A similar result is Lemma 2.1(a) of Tang and Yang (2012).

Lemma 4.1 Let \( X \in L_{\phi}^{\infty} \). Define \( g(x) = E[\varphi((X - x)_{+})] \). Then, for every \( x \in \mathbb{R} \),

\[
g_{+}'(x) = -E[\varphi_{-}(X - x)1_{(X > x)}] \quad \text{and} \quad g_{-}'(x) = -E[\varphi_{+}(X - x)1_{(X > x)}].
\]

In particular, if the Young function \( \varphi(\cdot) \) be continuously differentiable over \( \mathbb{R}_{+} \), and either \( \varphi_{+}'(0) = 0 \) or \( F \) is continuous at \( x \), then \( g'(x) = -E[\varphi'(X - x)1_{(X > x)}] \).

The next lemma has a similar flavor and can be proved similarly. We show its proof here for completeness.

Lemma 4.2 Let \( X \in L_{\phi}^{\infty} \). Define \( g(s) = E[\varphi(sX_{+})] \). Then, for every \( s > 0 \),

\[
g_{+}'(s) = E[\varphi_{+}(sX_{+})X_{+}] \quad \text{and} \quad g_{-}'(s) = E[\varphi_{-}(sX_{+})X_{+}].
\]
Proof. If \( \Pr(X \leq 0) = 1 \), then both results are trivial. Therefore, we assume that \( X \) has a non-trivial positive part. In this case, denote by \( \tilde{X} \) the part of \( X \) restricted to \( X > 0 \); that is, \( \tilde{X} = X | X > 0 \). We have

\[
g(s) = E \left[ \varphi(sX) 1_{(X>0)} \right] = \overline{F}(0) E \left[ \varphi \left( s\tilde{X} \right) \right].
\]

Let \( \Delta s \) be small but not 0. By the convexity of \( \varphi(\cdot) \) over \( \mathbb{R}_+ \), if \( 0 < \Delta s \leq 1 \) then

\[
0 \leq \frac{g(s + \Delta s) - g(s)}{\Delta s} \leq \overline{F}(0) \left( E \left[ \varphi \left( (s + 1)\tilde{X} \right) \right] - E \left[ \varphi \left( s\tilde{X} \right) \right] \right) < \infty,
\]

while if \(-s/2 \leq \Delta s < 0\) then

\[
0 \leq \frac{g(s + \Delta s) - g(s)}{\Delta s} \leq \overline{F}(0) \left( E \left[ \varphi \left( s\tilde{X} \right) \right] - E \left[ \varphi \left( (s - s/2)\tilde{X} \right) \right] \right) < \infty.
\]

Thus, when taking \( \Delta s \downarrow 0 \) or \( \Delta s \uparrow 0 \) to

\[
\frac{g(s + \Delta s) - g(s)}{\Delta s} = E \left[ \frac{\varphi \left( (s + \Delta s)\tilde{X} \right) - \varphi \left( s\tilde{X} \right) \Delta s}{\Delta s} \right],
\]

we can apply the dominated convergence theorem to interchange the order of the limit and expectation. ■

Lemma 4.3 Let the Young function \( \varphi(\cdot) \) be strictly convex and continuously differentiable over \( \mathbb{R}_+ \), and let the random variable \( X \in L^\varphi_0 \). Let \( h = h(x) \) be an implicit function of \( x \in (-\infty, \hat{x}) \) determined by equation (1.1) with \( q \in (0, 1) \) fixed. Then \( h(x) \) is continuously differentiable over \((-\infty, \hat{x})\) if \( \varphi'(0) = 0 \) or \( F \) is continuous.

Proof. We apply the well-known implicit function theorem. Denote the LHS of (1.1) by \( g(x, h) \); namely,

\[
g(x, h) = E \left[ \varphi \left( \frac{(X - x)_+}{h} \right) \right].
\]

It suffices to check that, over \((-\infty, \hat{x}) \times (0, \infty)\), the bivariate function \( g(x, h) \) is continuously differentiable with respect to both \( x \) and \( h \) and that \( \partial g/\partial h \neq 0 \). By Lemmas 4.1 and 4.2, for \((x, h) \in (-\infty, \hat{x}) \times (0, \infty)\) we have

\[
\frac{\partial}{\partial x} g(x, h) = -\frac{1}{h} E \left[ \frac{\varphi'(X - x)}{h^2} 1_{(X>x)} \right], \tag{4.1}
\]

and

\[
\frac{\partial}{\partial h} g(x, h) = -E \left[ \varphi'(X - x) \frac{X - x}{h^2} 1_{(X>x)} \right]. \tag{4.2}
\]

Since \( \varphi'(\cdot) \) is continuous and positive over \((0, \infty)\), it is easy to see that both \( \partial g/\partial x \) and \( \partial g/\partial h \) obtained above are continuous in both \( x \) and \( h \) and that \( \partial g/\partial h \) is negative over \((-\infty, \hat{x}) \times (0, \infty)\). ■
5 The Limit of the Orlicz Quantile

Through the system of equations (1.1) and (3.2), both \( x^\ast \) and \( h^\ast \) are functions of \( q \). The next lemma reveals a general limit behavior for the pair \((x^\ast, h^\ast)\) appearing in (3.1) as \( q \uparrow 1 \).

**Lemma 5.1** Let the Young function \( \varphi(\cdot) \) be convex over \( \mathbb{R}_+ \) and let the random variable \( X \in L^\varphi_\alpha \). Then, as \( q \uparrow 1 \), it holds for every \( \delta > 0 \) that \( x^\ast + \delta h^\ast \to \hat{x} \).

**Proof.** Let \( 0 < \delta < 1 \) be arbitrarily fixed (hence, \( 0 < \varphi(\delta) < 1 \)). Define \( \tilde{\varphi}(s) = \varphi(\delta s) / \varphi(\delta) \), which is still a Young function. Since \((x^\ast, h^\ast)\) solves (1.1), we have

$$
E \left[ \tilde{\varphi} \left( \frac{(X - x^\ast)_+}{\delta h^\ast} \right) \right] = \frac{1 - q}{\varphi(\delta)}.
$$

Let \( q \) be close to 1 such that \( 0 < (1 - q) / \varphi(\delta) < 1 \). Therefore, by Lemma 3.1 and Theorem 3.1 of Goovaerts et al. (2004), we have

$$
F^\leftarrow \left( 1 - \frac{1 - q}{\varphi(\delta)} \right) \leq x^\ast + \delta h^\ast \leq x^\ast + h^\ast \leq \hat{x}.
$$

The conclusion follows immediately. ■

An implication of Lemma 5.1 is that, when \( \hat{x} < \infty \), as \( q \uparrow 1 \) we have \( x^\ast \uparrow \hat{x} \) and \( h^\ast \downarrow 0 \). Therefore, in the theorem below we consider \( \hat{x} = \infty \) only.

**Theorem 5.1** In addition to the conditions of Theorem 3.1, assume that \( \varphi(\cdot) \in RV_\alpha(0+) \cap RV_\beta(\infty) \) for some \( 1 < \alpha, \beta < \infty \) and that \( X \) satisfies \( E[X^p] < \infty \) for some \( p > \alpha \vee \beta \). Then \( x^\ast \uparrow \infty \) as \( q \uparrow 1 \).

**Proof.** Assume by contradiction that \( x^\ast \) does not diverge to \( \infty \) as \( q \uparrow 1 \). Then there is a subsequence of \( q \uparrow 1 \), still denoted by \( q \uparrow 1 \) for simplicity, along which \( x^\ast \to x_0 \in [-\infty, \infty) \). By Lemma 5.1 we have \( h^\ast \uparrow \infty \) as \( q \uparrow 1 \). We are going to construct a contradiction based on (3.2).

**Case 1.** Suppose that \( x^\ast \to x_0 \in (-\infty, \infty) \). Since \( \varphi(\cdot) \in RV_\alpha(0+) \cap RV_\beta(\infty) \) is continuously differentiable, by Theorems 1.7.2 and 1.7.2b of Bingham et al. (1987) we have \( \varphi'(\cdot) \in RV_{\alpha-1}(0+) \cap RV_{\beta-1}(\infty) \). Therefore by Lemma 2.1, for arbitrarily small \( \varepsilon > 0 \), there exists some \( \delta > 0 \) such that, for all \( 0 < s, t < \delta \),

$$
(1 - \varepsilon) \left( \left( \frac{t}{s} \right)^{\alpha-1+\varepsilon} \wedge \left( \frac{t}{s} \right)^{\alpha-1-\varepsilon} \right) \leq \frac{\varphi'(t)}{\varphi(s)} \leq (1 + \varepsilon) \left( \left( \frac{t}{s} \right)^{\alpha-1+\varepsilon} \vee \left( \frac{t}{s} \right)^{\alpha-1-\varepsilon} \right). \quad (5.1)
$$

Let the inequalities in (5.1) and Lemma 5.1 share the same \( \delta \). Split the LHS of (3.2) into two parts as

$$
\text{LHS of (3.2)} = E \left[ \varphi' \left( \frac{(X - x^\ast)_+}{h^\ast} \right) \left( 1 \left( \frac{(X - x^\ast)_+}{h^\ast} \leq \delta \right) + 1 \left( \frac{(X - x^\ast)_+}{h^\ast} > \delta \right) \right) \right] = I_1(x^\ast, h^\ast) + I_2(x^\ast, h^\ast). \quad (5.2)
$$
By the second inequality in (5.1) we have
\[
\frac{I_1(x_*, h_*)}{\varphi'\left(\frac{1}{h_*}\right)} \leq (1 + \varepsilon) E \left[ (X - x_*)_+^{\alpha - 1 + \varepsilon} \vee (X - x_*)_+^{\alpha - 1 - \varepsilon} 1_{(X \leq x_* + \delta h_*)} \right].
\]

Thus, by the dominated convergence theorem and Lemma 5.1, we have
\[
\lim_{q \uparrow 1} \frac{I_1(x_*, h_*)}{\varphi'\left(\frac{1}{h_*}\right)} = E \left[ \lim_{q \uparrow 1} \frac{\varphi\left(\frac{(X - x_*)_+}{h_*}\right)}{\varphi'\left(\frac{1}{h_*}\right)} 1_{\left(\frac{(X - x_*)_+}{h_*}\right) \leq \delta} \right] = E \left[ (X - x_0)_+^{\alpha - 1} \right].
\]

For \( I_2(x_*, h_*) \), by (2.1) we have
\[
\left\{ \begin{array}{ll}
\varphi'(s) \geq C s^{\alpha - 1 + \varepsilon}, & \text{for } 0 < s \leq \delta, \\
\varphi'(s) \leq C s^{\beta - 1 - \varepsilon} \leq C_s^{(\alpha \vee \beta) - 1 + \varepsilon}, & \text{for } \delta \leq s < \infty.
\end{array} \right. \tag{5.3}
\]

As \( q \uparrow 1 \), since \( x_* \) is bounded, \( h_* \uparrow \infty \), and \( E[X^n_+] < \infty \), by (5.3) we have
\[
I_2(x_*, h_*) \leq CE \left[ \left(\frac{(X - x_*)_+}{h_*}\right)^{(\alpha \vee \beta) - 1 + \varepsilon} 1_{(X > x_* + \delta h_*)} \right] = \frac{o(1)}{h_*^{(\alpha \vee \beta) - 1 + \varepsilon}} = o \left( I_1(x_*, h_*) \right),
\]

where in the second step we used Lemma 5.1. It follows from (5.2) that
\[
\text{LHS of (3.2)} \sim \varphi'\left(\frac{1}{h_*}\right) E \left[ (X - x_0)_+^{\alpha - 1} \right].
\]

Since the function \( \psi(\cdot) \) defined by \( \psi(s) = s \varphi'(s) \) for \( s \geq 0 \) belongs to \( \text{RV}_\alpha(0+) \cap \text{RV}_\beta(\infty) \), analogously, we can obtain that
\[
\text{RHS of (3.2)} \sim \frac{1}{h_*} \varphi'\left(\frac{1}{h_*}\right) E \left[ (X - x_0)_+^{\alpha} \right].
\]

A combination of these estimates for both sides of (3.2) yields a contradiction.

**Case 2.** Suppose that \( x_* \downarrow -\infty \) as \( q \uparrow 1 \). By Lemma 5.1, \( x_* = o(h_*) \) as \( q \uparrow 1 \). For the LHS of (3.2), we still use the decomposition in (5.2). For arbitrarily small \( \varepsilon > 0 \), we have
\[
I_1(x_*, h_*) \geq CE \left[ \left(\frac{(X - x_*)_+}{h_*}\right)^{\alpha - 1 + \varepsilon} 1_{\left(\frac{(X - x_*)_+}{h_*}\right) \leq \delta} \right] \\
= C |x_*|^{\alpha - 1 + \varepsilon} h_*^{-1 + \varepsilon} E \left[ \left(\frac{(X - x_*)_+}{|x_*|^{\alpha - 1 + \varepsilon}}\right) 1_{(X \leq x_* + \delta h_*)} \right] \\
\sim C |x_*|^{\alpha - 1 + \varepsilon} h_*^{-1 + \varepsilon},
\]

where in the first step we applied the first inequality in (5.3), while in the last step we applied the dominated convergence theorem to prove that the expectation converges to 1 as \( x_* \downarrow -\infty \). The other asymptotic bound for \( I_1(x_*, h_*) \) can also be established. Thus,
\[
C |x_*|^{\alpha - 1 + \varepsilon} h_*^{-1 + \varepsilon} \lesssim I_1(x_*, h_*) \lesssim C |x_*|^{\alpha - 1 - \varepsilon} h_*^{\alpha - 1 - \varepsilon}.
\]
Now we turn to $I_2(x_*, h_*)$. Applying (5.3), $c_r$-inequality and Markov’s inequality, in turn, we have

\[
I_2(x_*, h_*) \leq C \mathbb{E} \left[ \left( \frac{(X - x_*)^+}{h_*} \right)^{\beta - 1 + \varepsilon} 1_{(X > x_*)} \right]
\]

\[
\leq \frac{C}{h_*^{\beta - 1 + \varepsilon}} \mathbb{E} \left[ X_+^{\beta - 1 + \varepsilon} + |x_*|^{\beta - 1 + \varepsilon} \right] 1_{(X > x_*)}
\]

\[
\leq \frac{C}{h_*^{\beta - 1 + \varepsilon}} \mathbb{E} \left[ X_+^{\beta - 1 + \varepsilon} \frac{X_+^{(\alpha \vee \beta) - \beta + 1}}{(x_* + \delta h_*)^{(\alpha \vee \beta) - \beta + 1}} + |x_*|^{\beta - 1 + \varepsilon} \frac{X_+^{(\alpha \vee \beta) + \varepsilon}}{(x_* + \delta h_*)^{(\alpha \vee \beta) + \varepsilon}} \right]
\]

\[
= \frac{C}{h_*^{\beta - 1 + \varepsilon}} \left( \frac{1}{(x_* + \delta h_*)^{(\alpha \vee \beta) - \beta + 1}} + |x_*|^{\beta - 1 + \varepsilon} \frac{1}{(x_* + \delta h_*)^{(\alpha \vee \beta) + \varepsilon}} \right) \mathbb{E} \left[ X_+^{(\alpha \vee \beta) + \varepsilon} \right]
\]

where the last step is due to the fact that $x_* = o(h_*)$ as $q \uparrow 1$. Since $x_* \downarrow -\infty$ and $h_* \uparrow \infty$, we have $I_2(x_*, h_*) = o(I_1(x_*, h_*))$. It follows from (5.2) that

\[
C \frac{|x_*|^{\alpha - 1 + \varepsilon}}{h_*^{\alpha - 1 + \varepsilon}} \lesssim \text{LHS of (3.2)} \lesssim C \frac{|x_*|^{\alpha - 1 - \varepsilon}}{h_*^{\alpha - 1 - \varepsilon}}.
\]

Analogously,

\[
C \frac{|x_*|^{\alpha + \varepsilon}}{h_*^{\alpha + \varepsilon}} \lesssim \text{RHS of (3.2)} \lesssim C \frac{|x_*|^{\alpha - \varepsilon}}{h_*^{\alpha - \varepsilon}}.
\]

Since $x_* = o(h_*)$, a combination of these estimates for both sides of (3.2) yields a contradiction.  

\section{A Deeper Description for the Orlicz Quantile}

Recall the function $U(\cdot)$ defined in Subsection 2.4 and let $V$ be a uniform $(0, 1)$ distributed random variable. By setting $t_* = 1/F(x_*)$, we have $U(t_*) \leq x_* \leq U(t_* + 0)$. Note that, by Lemma 5.1 and Theorem 5.1, we have $t_* \uparrow \infty$ as $q \uparrow 1$. It is easy to verify that

\[
X| (X > x_*) \overset{d}{=} U(t_*/V).
\]

Hence,

\[
\mathbb{E} \left[ \varphi \left( \frac{(X - x_*)^+}{h_*} \right) \right] = \frac{1}{t_*} \mathbb{E} \left[ \varphi \left( \frac{X - x_*}{h_*} \right) \right] 1_{(X > x_*)} \leq \frac{1}{t_*} \mathbb{E} \left[ \varphi \left( \frac{U(t_*/V) - U(t_*)}{h_*} \right) \right].
\]

Similarly, a lower bound can be established as

\[
\mathbb{E} \left[ \varphi \left( \frac{(X - x_*)^+}{h_*} \right) \right] \geq \frac{1}{t_*} \mathbb{E} \left[ \varphi \left( \frac{U(t_*/V) - U(t_* + 0)}{h_*} \right) \right].
\]

The idea in these derivations will be often used in this sections.
Lemma 6.1 Under the conditions of Theorem 5.1, the auxiliary function \( a(\cdot) \) in (2.2) satisfies \( a(t_*) \asymp h_* \) as \( q \uparrow 1 \).

**Proof.** Our proof is still based on equation (3.2). First assume by contradiction that there exists a subsequence of \( q \uparrow 1 \), still denoted by \( q \uparrow 1 \) for simplicity, along which \( a(t_*)/h_* \) diverges to \( \infty \). By the idea mentioned in the beginning of this section, we can derive a similar upper bound for the LHS of (3.2) and a similar lower bound for the RHS of (3.2), so that

\[
\int_0^1 \varphi' \left( \frac{U(t_*/v) - U(t_* + 0)}{h_*} \right) \frac{U(t_*/v) - U(t_* + 0)}{h_*} dv \leq \int_0^1 \varphi' \left( \frac{U(t_*/v) - U(t_*)}{h_*} \right) dv.
\]

By Lemma 2.3, it holds for every small \( \varepsilon, \delta > 0 \) and all large \( t_* \) and all \( 0 < v < 1 \) that

\[
\left| \frac{U(t_*/v) - U(t_*)}{a(t_*)} - \frac{v^{-\gamma} - 1}{\gamma} \right| \leq \varepsilon v^{-\gamma - \delta}.
\]

By Lemma 2.1 and \( \varphi'(\cdot) \in RV_{\beta-1}(\infty) \), for arbitrarily fixed small \( 0 < \varepsilon < 1 \), there is some \( M > 0 \) such that, for all \( s, t > M \),

\[
\frac{\varphi'(t)}{\varphi'(s)} \leq (1 + \varepsilon) \left( \frac{t}{s} \right)^{\beta - 1 + \varepsilon} \left( \frac{t}{s} \right)^{\beta - 1 - \varepsilon}.
\]

Let (6.2) and (6.3) share the same \( \varepsilon \). By these two inequalities we have

\[
\frac{\text{RHS of (6.1)}}{\varphi'(a(t_*)/h_*)} = \int_0^1 \frac{\varphi' \left( \frac{U(t_*/v) - U(t_*)}{a(t_*)/h_*} \right)}{\varphi' \left( \frac{a(t_*)}{h_*} \right)} dv
\]

\[
\leq \int_0^1 \frac{\varphi' \left( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \frac{a(t_*)}{h_*} \right)}{\varphi' \left( \frac{a(t_*)}{h_*} \right)} dv
\]

\[
\leq C \int_0^1 \left( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \right)^{\beta - 1 + \varepsilon} dv \leq \infty
\]

since \( \gamma < \alpha^{-1} \wedge \beta^{-1} \). An application of dominated convergence theorem yields that, as \( q \uparrow 1 \),

\[
\text{RHS of (6.1)} \sim \varphi' \left( \frac{a(t_*)}{h_*} \right) \int_0^1 \left( \frac{v^{-\gamma} - 1}{\gamma} \right)^{\beta - 1} dv.
\]

Analogously, as \( q \uparrow 1 \),

\[
\text{LHS of (6.1)} \sim \varphi' \left( \frac{a(t_*)}{h_*} \right) a(t_*) \frac{a(t_*)}{h_*} \int_0^1 \left( \frac{v^{-\gamma} - 1}{\gamma} \right)^{\beta} dv.
\]

Due to the assumption that \( a(t_*)/h_* \to \infty \) as \( q \uparrow 1 \), a combination of these two relations contradicts to (6.1).
Next assume by contradiction that there exists a subsequence of \( q \uparrow 1 \), still denoted by \( q \uparrow 1 \) for simplicity, along which \( a(t_*)/h_* \) converges to 0. Analogously, we derive an lower bound for the LHS of (3.2) and an upper bound for the RHS of (3.2) and then we work on the inequality
\[
\int_0^1 \varphi' \left( \frac{U(t_*/v) - U(t_*/0)}{h_*} \right) dv \leq \int_0^1 \varphi' \left( \frac{U(t_*/v) - U(t_*)}{h_*} \right) \frac{U(t_*/v) - U(t_*)}{h_*} dv. \tag{6.4}
\]
We deal with the LHS of (6.4) and aim to apply the dominated convergence theorem to obtain that
\[
\lim_{q \uparrow 1} \frac{\text{LHS of (6.4)}}{\varphi' \left( \frac{a(t_*)}{h_*} \right)} = \lim_{q \uparrow 1} \int_0^1 \varphi' \left( \frac{U(t_*/v) - U(t_*/0)}{a(t_*)} \frac{a(t_*)}{h_*} \right) dv = \int_0^1 \lim_{q \uparrow 1} \varphi' \left( \frac{U(t_*/v) - U(t_*/0)}{a(t_*)} \frac{a(t_*)}{h_*} \right) dv = \int_0^1 \left( \frac{v^{-\gamma} - 1}{\gamma} \right)^{\alpha - 1} dv. \tag{6.5}
\]
To validate the derivation in (6.5), we verify the requirement for the dominated convergence theorem. Similarly to the above, using (6.2) we have
\[
\text{LHS of (6.4)} \leq \int_0^1 \varphi' \left( \left( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \right) \frac{a(t_*)}{h_*} \right) dv.
\]
By Lemma 2.1 and \( \varphi'(*) \in RV_{\alpha - 1}(0+) \), for arbitrarily fixed small \( 0 < \varepsilon < 1 \), there is some small \( \delta_1 > 0 \) such that (5.1) holds for \( 0 < s, t < \delta_1 \). For all \( q \) close to 1 we have \( a(t_*)/h_* \leq \delta_1 \). Then define
\[
v_0 = \inf \left\{ v \in (0, 1] : \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \frac{a(t_*)}{h_*} \leq \delta_1 \right\},
\]
which varies in \( a(t_*)/h_* \). If \( \gamma < 0 \), then the function \( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \) is bounded over \([0, 1]\). Thus, when \( a(t_*)/h_* \) becomes small enough, the set in the definition of \( v_0 \) coincides with \((0, 1]\) and \( v_0 = 0 \). On the other hand, if \( \gamma \geq 0 \), the function \( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \) decreases in \( v \in (0, 1] \) and \( v_0 \) must be strictly positive, despite that \( v_0 \) tends to 0 as \( a(t_*)/h_* \) becomes small. In any case we assume that \( a(t_*)/h_* \) is small enough such that \( 0 \leq v_0 < 1 \). In terms of this \( v_0 \) we do the split
\[
\text{LHS of (6.4)} \leq \left( \int_0^{v_0} + \int_{v_0}^1 \right) \varphi' \left( \left( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \right) \frac{a(t_*)}{h_*} \right) dv = I_1 + I_2.
\]
By (5.3),
\[
I_1 \leq C \int_0^{v_0} \left( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \right) \frac{a(t_*)}{h_*} \left( \alpha \wedge \beta \right)^{-1 + \varepsilon} dv
= o(1) \varphi' \left( \frac{a(t_*)}{h_*} \right) \int_0^{v_0} \left( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \right) \left( \alpha \wedge \beta \right)^{-1 + \varepsilon} dv.
\]
By the second inequality in (5.1),
\[ I_2 \leq (1 + \varepsilon) \varphi \left( \frac{a(t_*)}{h_*} \right) \int_0^1 \left( \left( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \right)^{a-1+\varepsilon} \vee \left( \frac{v^{-\gamma} - 1}{\gamma} + \varepsilon v^{-\gamma - \delta} \right)^{a-1-\varepsilon} \right) dv. \]

Note that the two integrals in the bounds above for \( I_1 \) and \( I_2 \) are finite since \( \gamma < \alpha^{-1} \wedge \beta^{-1} \). This validates the application of the dominated convergence theorem in (6.5) and, hence,

**LHS of (6.4)**
\[ \sim \varphi \left( \frac{a(t_*)}{h_*} \right) \int_0^1 \left( \frac{v^{-\gamma} - 1}{\gamma} \right)^{a-1} dv. \]

Analogously,

**RHS of (6.4)**
\[ \sim \varphi \left( \frac{a(t_*)}{h_*} \right) \frac{a(t_*)}{h_*} \int_0^1 \left( \frac{v^{-\gamma} - 1}{\gamma} \right)^{a} dv. \]

A combination of these two relations contradicts to (6.4). □

**Lemma 6.2** Under the conditions in Theorem 3.2, equation (3.6) has a unique positive solution.

**Proof.** First prove the existence. Define
\[ g(\lambda) = \frac{\mathbb{E}[\varphi'(\lambda Y) \lambda Y]}{\mathbb{E}[\varphi'(\lambda Y)]}, \]
which is obviously a continuous function over \( \lambda > 0 \). Thus, as \( \lambda \downarrow 0 \), by the second inequality in (5.1) and (5.3),
\[ \frac{\varphi'(\lambda Y)}{\varphi'(\lambda)} = \frac{\varphi'(\lambda Y)}{\varphi'(\lambda)} \left( 1_{\{\lambda Y \leq \delta\}} + 1_{\{\lambda Y > \delta\}} \right) \leq (1 + \varepsilon) \left( Y^{a-1+\varepsilon} \vee Y^{a-1-\varepsilon} \right) + C \frac{(\lambda Y)^{(a \vee \beta)-1+\varepsilon}}{\lambda^{a-1+\varepsilon}}, \]
which is integrable since \( \gamma < \alpha^{-1} \wedge \beta^{-1} \). An application of the dominated convergence theorem gives that \( \mathbb{E}[\varphi'(\lambda Y)] \sim \varphi'(\lambda) \mathbb{E}[Y^\alpha] \) as \( \lambda \downarrow 0 \). Since the function \( \lambda \varphi'(\lambda) \) belongs to \( \text{RV}_\alpha(0+) \cap \text{RV}_\beta(\infty) \), we can also obtain \( \mathbb{E}[\varphi'(\lambda Y) \lambda Y] \sim \lambda \varphi'(\lambda) \mathbb{E}[Y^{\alpha}] \) as \( \lambda \downarrow 0 \). It follows that
\[ \lim_{\lambda \downarrow 0} g(\lambda) = 0. \]
Similarly,
\[ \lim_{\lambda \uparrow \infty} g(\lambda) = \infty. \]
This proves the existence of a solution to equation (3.6).
To prove the uniqueness of the solution, it suffices to show that $g'(\lambda) > 0$. First assume $\gamma > -1$. Note that

$$E[\varphi'(\lambda Y)] = -\int_0^{\hat{y}} \varphi'(\lambda y) d(1 + \gamma y)^{-1/\gamma} = \int_0^{\hat{y}} \left(1 + \frac{\gamma}{\lambda} w\right)^{-1/\gamma-1} d\varphi(w),$$

where $\hat{y}$ denotes the upper endpoint of $Y$, equal to $\infty$ if $\gamma \geq 0$ or $-1/\gamma$ if $\gamma < 0$. A similar expression for $E[\varphi'(\lambda Y \lambda Y)]$ can also be derived. Then we have

$$g'(\lambda) = \frac{(1 + \gamma) \lambda^{-2}}{(\int_0^{\lambda \hat{y}} (1 + \frac{\gamma}{\lambda} w)^{-1/\gamma-1} d\varphi(w))^2} \times \left[\int_0^{\lambda \hat{y}} w^2 \left(1 + \frac{\gamma}{\lambda} w\right)^{-1/\gamma-2} d\varphi(w) \int_0^{\lambda \hat{y}} \left(1 + \frac{\gamma}{\lambda} w\right)^{-1/\gamma-1} d\varphi(w) - \int_0^{\lambda \hat{y}} w \left(1 + \frac{\gamma}{\lambda} w\right)^{-1/\gamma-1} d\varphi(w) \int_0^{\lambda \hat{y}} \left(1 + \frac{\gamma}{\lambda} w\right)^{-1/\gamma-2} d\varphi(w)\right].$$

Every integral in the above is finite for $\gamma > -1$. It remains to verify that the part in the square brackets is positive. Introduce a random variable $W$ with distribution

$$\Pr(W \in dw) = \frac{(1 + \frac{\gamma}{\lambda} w)^{-1/\gamma-1} d\varphi(w)}{(\int_0^{\lambda \hat{y}} (1 + \frac{\gamma}{\lambda} w)^{-1/\gamma-1} d\varphi(w))}, \quad 0 < w < \lambda \hat{y}.$$

What we need to verify becomes

$$E\left[\frac{W^2}{1 + \frac{2}{\lambda} W}\right] > E[W] E\left[\frac{W}{1 + \frac{2}{\lambda} W}\right]. \quad (6.6)$$

If $\gamma = 0$, then inequality (6.6) reduces to $E[W^2] > (E[W])^2$, which is ensured by Jensen’s inequality. Now consider $\gamma \neq 0$. Notice that

$$\frac{W}{1 + \frac{2}{\lambda} W} = \frac{\lambda}{\gamma} - \frac{\lambda}{\gamma(1 + \frac{2}{\lambda} W)}.$$

This helps simplify inequality (6.6) to

$$E\left[\frac{W}{\gamma(1 + \frac{2}{\lambda} W)}\right] < E[W] E\left[\frac{1}{\gamma(1 + \frac{2}{\lambda} W)}\right],$$

which is further equivalent to

$$\text{Cov}\left(W, \frac{1}{\gamma(1 + \frac{2}{\lambda} W)}\right) < 0. \quad (6.7)$$

By Höffding’s formula for covariance, we know that inequality (6.7) holds true since the two random variables inside are counter-comonotonic and are not degenerate. See Dhaene et al. (2002) for the definitions of comonotonicity and counter-comonotonicity.
Next assume $\gamma \leq -1$. We derive an alternative expression for the derivative of $g(\lambda)$ as
\[
g'(\lambda) = \frac{E[\varphi'(\lambda Y)] (E[\varphi''(\lambda Y) \lambda^2 Y] + E[\varphi'(\lambda Y) Y]) - E[\varphi'(\lambda Y) \lambda Y] E[\varphi''(\lambda Y) Y]}{(E[\varphi'(\lambda Y)])^2}.
\]
Note that all the terms on the LHS are well defined. We need to verify that the numerator is positive for every $\lambda > 0$. Introduce two random variables $W_1$ and $W_2$ with distributions
\[
\Pr(W_1 \in dw) = \frac{\varphi'(\lambda w)}{E[\varphi'(\lambda Y)]} \Pr(Y \in dw) \quad \text{and} \quad \Pr(W_2 \in dw) = \frac{\varphi''(\lambda w)}{E[\varphi''(\lambda Y)]} \Pr(Y \in dw)
\]
for $0 \leq w \leq -1/\gamma$. It remains to verify that
\[
\lambda E[W_2^2] + \frac{E[\varphi'(\lambda Y) Y]}{E[\varphi''(\lambda Y)]} > \lambda E[W_1] E[W_2]. \tag{6.8}
\]
Notice that
\[
\frac{E[\varphi'(\lambda Y) Y]}{E[\varphi''(\lambda Y)]} = \frac{E[\varphi'(\lambda Y)] E[\varphi'(\lambda Y)]}{E[\varphi'(\lambda Y)] E[\varphi''(\lambda Y)]} = \frac{E[W_1] \lambda \int_{0}^{-1/\gamma} (1 + \gamma y) (1 + \gamma y)^{-1/\gamma - 1} \varphi''(\lambda y) dy}{E[\varphi''(\lambda Y)]} = \lambda E[W_1] E[1 + \gamma W_2].
\]
By Jensen’s inequality,
\[
\text{LHS of (6.8)} = \lambda E[W_2^2] + \lambda E[W_1] E[1 + \gamma W_2] \geq \lambda (E[W_2])^2 + \lambda E[W_1] E[1 + \gamma W_2].
\]
Then inequality (6.8) follows if we can show that
\[
\lambda (E[W_2])^2 + \lambda E[W_1] E[1 + \gamma W_2] > \lambda E[W_1] E[W_2],
\]
or, equivalently,
\[
E[W_1] + \gamma E[W_1] E[W_2] > E[W_2] (E[W_1] - E[W_2]). \tag{6.9}
\]
Note that both $W_1$ and $W_2$ are non-degenerate positive random variables bounded by $-1/\gamma \leq 1$ and that the LHS of (6.9) is positive. Thus, if $E[W_1] - E[W_2] \leq 0$, then inequality (6.9) naturally holds. On the other hand, if $E[W_1] - E[W_2] > 0$, we lower $\gamma E[W_1]$ to $-1$ on the LHS while raise the first $E[W_2]$ to $1$ on the RHS and then we see that inequality (6.9) holds. ■

Now we are ready to show the main result of this section.

**Theorem 6.1** Under the assumptions of Theorem 3.2, we have $\lim_{q \to 1} a(t_*)/h_* = \lambda$, where $\lambda$ is the unique solution to equation (3.6).
Proof. Consider a subsequence of $q$, still denoted by $q$ for simplicity, along which $a(t_*)/h_* \to \lambda_*$ for some $\lambda_* \in (0, \infty)$. We still work on (3.2). By the idea mentioned in the beginning of this section,

$$\text{LHS of (3.2)} \leq \frac{1}{t_*} E \left[ \varphi' \left( \frac{U(t_*/V) - U(t_*) a(t_*)}{a(t_*)/h_*} \right) \right].$$

By Lemma 2.3, it holds for every small $\varepsilon, \delta > 0$ and all large $t_*$ that

$$\varphi' \left( \frac{U(t_*/V) - U(t_*) a(t_*)}{a(t_*)/h_*} \right) \leq \varphi' \left( \left( \varepsilon V^{-\gamma - \delta} + \frac{V^{-\gamma} - 1}{\gamma} \right) (\lambda_* + \varepsilon) \right),$$

which is integrable since $\varphi' (\cdot) \in RV_{\beta-1}(\infty)$ and $\gamma < \alpha^{-1} \wedge \beta^{-1}$. Hence, by the dominated convergence theorem and (3.5),

$$\text{LHS of (3.2)} = F(x_*) E \left[ \varphi' \left( \frac{X - x_* a(t_*)}{a(t_*)/h_*} \right) \bigg| X > x_* \right] \sim F(x_*) E [\varphi' (\lambda_* Y)].$$

Similarly,

$$\text{RHS of (3.2)} \sim \overline{F}(x_*) E [\varphi' (\lambda_* Y) \lambda_* Y].$$

By Lemma 6.2, we know $\lambda_*$ must be identical to the unique solution of equation (3.6).

7 Numerical Examples

In this section, we use $R$ to numerically examine the accuracy of the asymptotic formulas derived in Section 3. In order to compare the asymptotic results with the exact value of $H_q[X]$, we choose the Young function to be

$$\varphi(s) = \frac{1}{2} \left( s^{2.2} + s^{1.1} \right), \quad s \geq 0.$$  

Then (1.1) becomes

$$E \left[ \left( \frac{(X - x)_+}{h} \right)^{2.2} + \left( \frac{(X - x)_+}{h} \right)^{1.1} \right] = 2(1 - q).$$

By the quadratic formula we solve $h$ as

$$h = \left( \frac{E [(X - x)^{1.1}]}{4(1 - q)} + \sqrt{E [(X - x)_+^{1.1}]^2 + 8(1 - q)E [(X - x)_+^{2.2}]} \right)^{1/1.1}.$$  

(7.1)

Then from the equation $h'(x) = -1$ we can get the Orlicz quantile $x_*$. Plugging this $x_*$ into (7.1) gives the value of $h_*$. Therefore, the exact value of $H_q[X] = x_* + h_*$ is obtained. In each of the following three cases, we will compare the asymptotic value of $H_q[X]$ computed according to Corollary 3.1 with its exact value.
Example 7.1 (The Fréchet case) Assume that $F$ is a Pareto distribution given by

$$F(x) = 1 - \left(\frac{\theta}{x + \theta}\right)^\alpha, \quad x, \alpha, \theta > 0.$$ 

Thus $F \in \text{MDA}(G_\gamma)$ with $\gamma = 1/\alpha$ and $U(t) = \theta(t^\gamma - 1)$. In Figure 7.1, setting $\alpha = 2.4, 2.7$ and $\theta = 1$, we compare the asymptotic value of $H_q[X]$ with its exact value on the left and show their ratio on the right. We find that the ratio converges to 1 as $q \uparrow 1$ but the accuracy is not as good as in the power Young function case, so the second-order asymptotics may become necessary to improve the accuracy.

Figure 7.1 about here.

Example 7.2 (The Gumbel case) Assume that $F$ is a lognormal distribution given by

$$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right), \quad x > 0, -\infty < \mu < \infty, \sigma > 0,$$

where $\Phi(\cdot)$ denotes the standard normal distribution function. Note that

$$a(x) = \frac{\Phi(\sigma^{-1}(\ln x - \mu))\sigma x}{\phi(\sigma^{-1}(\ln x - \mu))},$$

where $\phi$ is the standard normal density function; see, e.g. page 150 of Embrechts et al. (1997). Thus $F \in \text{MDA}(\Lambda)$ with $\gamma = 0$. In Figure 7.2, setting $\mu = 2$ and $\sigma = 0.5$, we compare the asymptotic value of $H_q[X]$ with its exact value on the left and show their ratio on the right. Overall, the accuracy is very good.

Figure 7.2 about here.

Example 7.3 (The Weibull case) Assume that $F$ is a beta distribution with probability density function given by

$$f(x) = x^{a-1}(1-x)^{b-1} \frac{1}{B(a,b)}, \quad 0 < x < 1, a, b > 0.$$ 

Thus $F \in \text{MDA}(G_\gamma)$ with $\gamma = -1/b$. In Figure 7.3, setting $a = 2$ and $b = 6, 8$, we compare the asymptotic value of $H_q[X]$ with its exact value on the left and show their ratio on the right. We find that for both cases the ratio converges to 1 as $q \uparrow 1$ and the accuracy is very good.

Figure 7.3 about here.
References


Asymptotic

Graph 7.1 Pareto distribution
Asymptotic

Exact

Graph 7.2 Lognormal distribution
Haezendonck–Goovaerts risk measure

Asymptotic

Exact

Graph 7.3 Beta distribution