Systematic Mortality Risk: An Analysis of Guaranteed Lifetime Withdrawal Benefits in Variable Annuities

This version: 20 May 2013

Man Chung Fung\textsuperscript{a}, Katja Ignatieva\textsuperscript{b} \textsuperscript{1}, Michael Sherris\textsuperscript{c}

\textsuperscript{a}School of Risk and Actuarial Studies, University of New South Wales, Sydney, Australia (m.c.fung@unsw.edu.au)

\textsuperscript{b}School of Risk and Actuarial Studies, University of New South Wales, Sydney, Australia (k.ignatieva@unsw.edu.au)

\textsuperscript{c}CEPAR, School of Risk and Actuarial Studies, University of New South Wales, Sydney, Australia (m.sherris@unsw.edu.au)

JEL Classification: G22, G23, G13

Abstract

Guaranteed lifetime withdrawal benefits (GLWB) embedded in variable annuities have become an increasingly popular type of life annuity designed to cover systematic mortality risk while providing protection to policyholders from downside investment risk. This paper provides an extensive study of how different sets of financial and demographic parameters affect the fair guaranteed fee charged for a GLWB as well as the profit and loss distribution, using tractable equity and stochastic mortality models in a continuous time framework. We demonstrate the significance of parameter risk, model risk, as well as the systematic mortality risk component underlying the guarantee. We quantify how different levels of equity exposure chosen by the policyholder affect the exposure of the guarantee providers to systematic mortality risk. Finally, the effectiveness of a static hedge of systematic mortality risk is examined allowing for different levels of equity exposure.

Key words: variable annuity, guaranteed lifetime withdrawal benefits (GLWB), systematic mortality risk, parameter risk, model risk, static hedging

---

\textsuperscript{1}Corresponding author: School of Risk and Actuarial Studies, Australian School of Business, University of New South Wales, Sydney, NSW-2052, Australia. Email: k.ignatieva@unsw.edu.au; Tel: +61 2 9385 6810.
Variable annuities (VA), introduced in the 1970s in the US, have proven to be popular among investors and retirees especially in North America and Japan. VA are insurance contracts which allow policyholders to invest their retirement savings in mutual funds. They are attractive because policyholders gain exposure to the equity markets with benefits based on the performance of the underlying funds, along with return guarantees as well as tax advantages. In addition to the VA, policyholders can elect guarantees to provide minimum benefits with the payment of guarantee fees. Since the 1990s, two kinds of embedded guarantees have been offered in these policies. These include the Guaranteed Minimum Death Benefits (GMDB) and the Guaranteed Minimum Living Benefits (GMLB), see Hardy (2003) and Ledlie et al. (2008). There are four types of guarantee that fall into the category of GMLB:

- Guaranteed Minimum Accumulation Benefits (GMAB)
- Guaranteed Minimum Income Benefits (GMIB)
- Guaranteed Minimum Withdrawal Benefits (GMWB)
- Guaranteed Lifetime Withdrawal Benefits (GLWB)

According to LIMRA\(^2\), the election rates of GLWB, GMIB, GMAB and GMWB in the fourth quarter of 2011 in the US were 59\%, 26\%, 3\% and 2\%, respectively. The popularity of the GLWB can be attributed to its flexibility in that the policyholder can specify how much he/she wants to withdraw every year for the lifetime of the insured regardless of the performance of the investment, subject to a limited nominal yearly amount. After the death of the insured, any savings remaining in the account are returned to the insured’s beneficiary. GLWB provide a life annuity with flexible features compared to traditional life annuity contracts.

As a type of a life annuity with benefits linked to the equity market, insurers who offer a GLWB embedded in variable annuities are subject to equity risk, interest rate risk, withdrawal risk and, in particular, systematic mortality risk. Non-systematic, or idiosyncratic mortality risk, is diversifiable, whereas systematic mortality risk, or longevity risk, cannot be eliminated through diversification. Systematic mortality risk arises due to the stochastic, or unpredictable, nature of survival probabilities. Insurance providers offering GLWB face systematic mortality risk since the guarantee promises to the insured a stream of income for life, even if the investment account is depleted. Stochastic mortality models, often implemented in an intensity-based framework, e.g. Biffis (2005), are required to quantify this systematic mortality risk.

1.1 Motivation and Contribution

Following the global financial crisis in 2007-2008, VA providers have suffered unexpectedly large losses arising from the benefits provided by the different types of guarantee being too generous. High volatility of equity markets during this period

\(^2\) http://www.limra.com/default.aspx
resulted in unexpectedly high hedging costs and exacerbated the difficulty in managing the risks underlying the guarantees. Guarantees were inherently difficult to price and manage because of their long term nature. Along with the equity risk, guarantees offering lifetime income were exposed to systematic mortality risk from higher than expected mortality improvements. An important example of a guarantee product where the risks are not well understood is the GLWB.

Bacinello et al. (2011) set out the three steps required for a reliable risk management process. First is risk identification. For the benefit and the premium component of a guarantee, different sources of risk, whether hedgeable or not, need to be determined. The second step involves risk assessment. The long term nature and complex payout structure of these guarantees involves a range of risks. An assessment of the interactions between these risks is essential when considering an adequate risk management program. Risk assessment is also used to determine which sources of risk have the largest impact on the pricing and risk management of the guarantee. The final step is the risk management action. Risk management for these guarantees involves capital reserving and hedging (Hardy (2003) and Ledlie et al. (2008)). Selecting the appropriate hedge instruments and determining hedging frequency, in particular, static or dynamic hedging, need to be considered, along with constraints such as transaction costs and liquidity of the hedge instruments. A reliable risk management process is particularly important for GLWB since it is a long term contract, lasting on average for several decades, and involves many different risks. Due to its recent popularity, VA providers have large exposures that have grown with the GLWB as an underlying guarantee.

The significance of GLWB products in providing longevity insurance for individuals desiring a guaranteed income in retirement with equity exposure has been recognized. Despite this, the analysis of systematic mortality risk in GLWB, along with its interaction with other risks, has not been studied in detail. Studies have focussed on only a particular aspect of this risk. One of the earliest modeling frameworks that focuses on GLWB is discussed in Holz et al. (2007), who take into account policyholder’s behavior and different product features. Kling et al. (2011) focus on volatility risk. Piscopo and Haberman (2011) assess mortality risk in GLWB but not the other risks and their interactions. Bacinello et al. (2011) and Ngai and Sherris (2011) include GLWB but only as part of a specific study of pricing or hedging guarantees/annuities.

Papers that investigate the impact of systematic mortality risk on guarantees other than GLWB include Ballotta and Haberman (2006) who value guaranteed annuity options (GAOs) and provide sensitivity analysis taking into account the interest rate risk and the systematic mortality risk. Kling et al. (2012) offer an analysis similar to Ballotta and Haberman (2006), but considering both, GAOs and guaranteed minimal income benefits (GMIB), and taking into account different hedging strategies. Ziveyi et al. (2013) price European options on deferred insurance contracts including pure endowment and deferred life annuity (i.e. GAO) by solving analytically the pricing partial differential equation in a presence of systematic mortality risk.

4 For instance, uncertainty stemming from the behavior of policyholders is an unhedgeable risk, which requires insurance companies to assess policyholders general past behavior.
A detailed analysis of the impact of systematic mortality risk on valuation and hedging, as well as its interaction with other risks underlying the GLWB is required. These issues are the focus of the present paper. We apply the risk management process described above to analyze equity and systematic mortality risks underlying the GLWB, as well as their interactions. We use risk measures and the profit and loss (P&L) distribution in order to identify, assess and partially mitigate financial and demographic risks, for the GLWB guarantee. P&L analysis has proven to be useful in assessing systematic mortality risk in pension annuities, see Hari et al. (2008). We use the affine mortality model approach. Dahl and Moller (2006) considers a time inhomogeneous square root process, Schrager (2006) and Biffis (2005) propose multi-factor affine processes which capture the evolution of mortality intensity for all ages simultaneously. A concise survey of mortality modeling and the development of longevity market can be found in Cairns et al. (2008). Our aim is to adopt a tractable model given the complex nature of the product.

We quantify how different sets of financial and demographic parameters affect the guarantee fees. We assess their effects on the P&L distributions, if the parameters are different from the “true” parameters of the underlying dynamics. Parameter risk is shown to be a significant risk reflecting the long term nature of the GLWB, given the exposure for the lifetime of the insured in a portfolio. We consider model risk by considering the case where guarantee providers assume mortality to be deterministic when the mortality is stochastic in its nature.

The paper also considers the capital reserve required due to the lack of hedging instruments to hedge the longevity risk. The paper uses P&L analysis to show that different levels of equity exposure chosen by the insured have, in the presence of systematic mortality risk, a significant impact on the risk of the GLWB from the guarantee providers’ perspective. Finally, the effectiveness of a static hedge of systematic mortality risk is also assessed allowing for different levels of equity exposure.

The remainder of the paper is organized as follows. Section 2 outlines the general features of GLWB including identification of risks and additional product features, along with the benefit and the premium components of the guarantee. Section 3 specifies the continuous time models used for the underlying equity and systematic mortality risk. Section 4 evaluates the GLWB using two approaches, which are shown to be equivalent. It generalizes the pricing result of the GMWB in Kolkiewicz and Liu (2012) to the case of the GLWB. Sensitivity analysis is conducted in Section 5, which quantifies the sources of risk with the larger impact on pricing of GLWB. Section 6 analyzes the impact of different types of risk on the P&L distributions of the guarantee, without incorporating any hedging strategies. Different levels of equity exposure, parameter risk and model risk are considered. Section 7 studies the effectiveness of a static hedge of systematic mortality risk using S-forward’s taking into account different levels of equity exposure. Section 8 summarizes the results and provides further concluding remarks.
2 Features of GLWB

A variable annuity (VA) is an insurance product where a policyholder invests his or her retirement savings in a mutual fund in the form of an investment account managed by an insurance company. VAs are attractive because the investment enjoys certain tax benefits along with investment in the equity market. A VA with a GLWB rider guarantees payment of a capped amount of income that can be withdrawn from the account every year for the lifetime of the policyholder, even if the investment account is depleted before the policyholder dies. Any remaining value in the account will be returned to the policyholder’s beneficiary. In return, the policyholder is required to pay a guarantee fee every year, which is proportional to the investment account value.

The simplest type of a GLWB attached to a VA, which will be referred to as a plain GLWB, is described in a continuous time setting. Let $A(0)$ be the retirement savings invested in a variable annuity at time $t = 0$. For a plain GLWB, the guaranteed withdrawal amount, that is, the amount which is allowed to be withdrawn from the account per unit of time, is determined as $g \cdot A(0)$ where $g$ is called guaranteed withdrawal rate and $A(0)$ is considered as the withdrawal base. A typical annual value of $g$ is 5% for a policyholder aged 65 at $t = 0$. The policyholder has the freedom to decide how much to withdraw per unit of time. However, a penalty may arise if the withdrawal amount is higher than the guaranteed withdrawal amount allowed by the insurer. The guarantee fee paid by the policyholder per unit of time is determined as $\alpha_g \cdot A(t)$ where $\alpha_g$ is the guaranteed fee rate and $A(t)$ is the account value at time $t$.

From the insurer’s point of view, the liability and the fee structure of a plain GLWB can be described concisely by two scenarios which are captured in Fig. 1 and Fig. 2.

---

Fig. 1. Scenario 1 (liability for the insurer): policyholder dies at the age of 100 years, 20 years after the account value is depleted. Withdrawal rate is 5% per year.

For both scenarios we assume a static withdrawal strategy, where the policyholder receives exactly the guaranteed withdrawal amount per unit of time. An alternative strategy, proposed in Milevsky and Salisbury (2006), is a dynamic withdrawal, which leads to an optimal stopping problem that aims to maximize the value of the guarantee. In Fig. 1, the policyholder dies at the age of around 100 and the account value drops to zero when the policyholder is 80 years old. The insurance company receives...
guarantee fees during the period when the policyholder is aged from 65 to 80. As the account is depleted and the policyholder is still alive at the age of 80, the insurance company has to continue paying the guaranteed withdrawal amount during the period when the policyholder is aged from 80 to 100. In this scenario there is a liability for the insurer. Another scenario is captured in Fig. 2. Here the policyholder dies at the age of 100, however, there is still a positive amount left in the account at the time of death of the insured. In this scenario the insurance company receives the guarantee fees during the lifetime of the policyholder while there is no liability for the insurer.

A plain GLWB can be enriched by adding features such as a roll-up feature and a step-up (or ratchet) feature. A roll-up feature ensures that the guaranteed withdrawal amount will be increased at least by a fixed amount every year, given the policyholder has not yet started to withdraw money. A step-up feature allows the withdrawal base to increase at specified points in time if the current account value exceeds the previous value of the withdrawal base. There are deferred versions of the contract, as opposed to immediate withdrawals, where the policyholder can only withdraw money after the deferred period. Details on additional features embedded in GLWB can be found in e.g. Piscopo and Haberman (2011) and Kling et al. (2011).

From Fig. 1 and Fig. 2 and the discussion above, it is clear that GLWB providers are subject to three types of risk: financial risk, demographic risk and behavioral risk. Financial risk includes interest rate risk and equity risk. Demographic risk consists of non-systematic and systematic mortality risk. Different withdrawal behaviors of the policyholder, which may or may not be rational, represent behavioral risk.

In order to focus on systematic mortality risk and its interaction with equity risk, the following simplifying assumptions are made. We assume interest rate and equity volatility to be constant. The withdrawal rate is assumed to be static and there are no lapses. Payments are immediate (not deferred), that is, the withdrawals start immediately and there is no deferment period for investment growth. We do not consider possible roll-up or step-up options, but note that the results presented in

---

5 A 100% withdrawal can be considered as lapse risk.
the following sections can be generalized when additional features are incorporated in the contract.

3 Model Specification

We base our analysis on tractable continuous-time models to capture equity and systematic mortality risk underlying GLWB in variable annuities. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a filtered probability space where \(\mathbb{P}\) is the real world probability measure. The filtration \(\mathcal{F}_t\) is constructed as

\[
\mathcal{G}_t = \sigma\{(W_1(s), W_2(s)) : 0 \leq s \leq t\}
\]

\[
\mathcal{H}_t = \sigma\{1_{\{\hat{\tau} \leq s\}} : 0 \leq s \leq t\}
\]

\[
\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t
\]

where \(\mathcal{G}_t\) is generated by two independent standard Brownian motions, \(W_1\) and \(W_2\), which are associated with the uncertainties related to equity and mortality intensity, respectively. The subfiltration \(\mathcal{H}_t\) represents information set that would indicate whether the death of a policyholder has occurred before time \(t\). The stopping time \(\hat{\tau}\) is interpreted as the remaining lifetime of an insured. For a detailed exposition of modeling mortality under the intensity-based framework refer to Biffis (2005).

3.1 Investment Account Dynamics

The policyholder can invest his or her retirement savings in an investment fund that has both, equity and fixed income exposure. Under the real world probability measure \(\mathbb{P}\), we assume that the equity component follows the geometric Brownian motion

\[
dS(t) = \mu S(t) dt + \sigma S(t) dW_1(t),
\]

and the fixed income investment is represented by the money market account \(B(t)\) with dynamics \(dB(t) = r B(t) dt\) where the interest rate \(r\) is constant. Let \(\delta_1(t)\) and \(\delta_2(t)\) denote the number of units invested in \(S(\cdot)\) and \(B(\cdot)\), respectively. The self-financing investment fund \(V(\cdot)\) has the following dynamics

\[
dV(t) = \delta_1(t) dS(t) + \delta_2(t) dB(t)
\]

\[
= (\mu \delta_1(t) S(t) + r \delta_2(t) B(t)) dt + \sigma \delta_1(t) S(t) dW_1(t)
\]

\[
= (\mu \pi(t) + r (1 - \pi(t))) V(t) dt + \sigma \pi(t) V(t) dW_1(t)
\]

where the fraction \(\pi(\cdot)\) is defined as \(\pi(t) = \frac{\delta_1(t) S(t)}{V(t)}\), and is interpreted as the proportion of the retirement savings being invested in the equity component. Since short-selling is not allowed, we set \(0 \leq \pi(\cdot) \leq 1\).

Because of the election of GLWB, the investment account \(A(\cdot)\) held by the policyholder is being charged continuously with guarantee fees, denoted by fee rate \(\alpha_g\), by
the insurer. Typically, the guarantee fee charged is proportional to the investment account $A(\cdot)$. We consider the case that a policyholder withdraws a constant amount continuously until death and the withdrawal amount is proportional to the guarantee base, assumed to be the initial investment $A(0)$. We denote the (static) guarantee withdrawal rate by $g$, which is so-named because the withdrawals are guaranteed by the insurer regardless of the investment performance, and define $G = g \cdot A(0)$ to be the withdrawal amount per unit of time.

The investment account of the policyholder satisfies the following dynamics:

$$dA(t) = \left(\mu \pi(t) + r(1 - \pi(t)) - \alpha_g\right)A(t)dt - Gdt + \sigma \pi(t)A(t)dW_1(t) \quad (3.3)$$

with $A(\cdot) \geq 0$, since the investment account value cannot be negative. In the following, we assume that $\pi(\cdot)$ is constant, that is, the policyholder invests in a fixed proportion of his/her retirement savings in the equity and fixed income markets throughout the investment period. In particular, we consider five cases where $\pi \in \{0, 0.3, 0.5, 0.7, 1\}$. Each value corresponds to a specific risk-preference of the retiree where a larger value of $\pi$ indicates that the insured prefers to have a larger equity exposure in his/her savings, which would result in a higher potential growth but will be subject to higher volatility as well. By allowing different levels of equity exposure, we will be able to study the interaction between equity and systematic mortality risk due to the specific design of the product.

3.2 Mortality Model

The mortality model is selected based on the following three criteria. First of all, the model should be qualitatively reasonable. This means that the mortality intensity should be strictly positive. Moreover, as argued in Cairns et al. (2006) and Cairns et al. (2008), it is unreasonable to assume that the mortality intensity is mean reverting, even if the mean level is time dependent. The second criterion is tractability. Given the complicated payout structure, a tractable stochastic mortality model is required for efficient pricing and risk management of the GLWB. The model should allow analytical expressions for important quantities, such as survival probabilities. It should also be computationally efficient, in the sense that one should be able to simulate the derived solutions paths of the underlying dynamics quickly and accurately. Finally, the ability of the model to be reduced to a simple deterministic mortality model is an appealing feature, as it allows simple investigation of the effect of, and the difference between, systematic and non-systematic mortality risk underlying the guarantee.

Given these criteria, we adopt a one-factor, non mean-reverting and time homogeneous affine process for modeling the mortality intensity process $\mu_{x+t}(t)$ of a person.

---

6 Mean reverting behavior of mortality intensity would indicate that if mortality improvements have been faster than anticipated in the past then the potential for further mortality improvements will be significantly lower in the future.
aged $x$ at time $t = 0$, as follows:

$$d\mu_{x+t}(t) = (a + b\mu_{x+t}(t))dt + \sigma_\mu \sqrt{\mu_{x+t}(t)} \, dW_2(t), \quad \mu_x(0) > 0. \quad (3.4)$$

Here, $a \neq 0$, $b > 0$ and $\sigma_\mu$ represents the volatility of the mortality intensity.

In the special case when $\sigma_\mu = a = 0$, the model reduces to the well-known constant Gompertz specification.\(^7\) Hence, we will be able to conveniently compare the impact of stochastic mortality with that of deterministic mortality in Sec. 6.3. The values of the parameters $a$, $b$ and $\sigma_\mu$ are obtained by calibrating the survival curve implied by the mortality model to the survival curve obtained from population data as documented in the Australian Life Tables 2005-2007\(^8\), for a male aged 65. The result is reported in Table 1.

**Table 1**

Calibrated parameters for the mortality model.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$\sigma_\mu$</th>
<th>$\mu_{65}(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.087</td>
<td>0.021</td>
<td>0.01147</td>
</tr>
</tbody>
</table>

The estimated values of the parameters $a$ and $\sigma_\mu$ indicate that the proposed mortality intensity process in Eq.(3.4) is strictly positive\(^9\) and is non mean-reverting. Modeling mortality intensity using a one-factor and time homogeneous affine process can reduce computational time significantly, since the model dynamics can be easily simulated and used to derive analytical expressions for survival probabilities. Thus, the proposed model in Eq.(3.4) satisfies all criteria stated above.

Because of the affine and time homogeneous assumption, we have an analytical expression for the survival probability $sP_{x+t}$ for the remaining lifetime $\hat{\tau}$ of an individual aged $x$ at time 0. Assuming that the individual is still alive at $t > 0$ and that $\sigma_\mu > 0$, we obtain

$$sP_{x+t} = E_t^Q \left( e^{-\int_{t}^{t+\hat{\tau}} \mu_{x+v} \, dv} \right) = C_1(s)e^{-C_2(s)\mu_{x+t}(t)}, \quad (3.5)$$

where

$$C_1(s) = \left( \frac{2\gamma e^{\frac{1}{2}(\gamma-b)s}}{(\gamma-b)(e^{\gamma s} - 1) + 2\gamma} \right)^{\frac{a}{\sigma_\mu}} \quad , \quad C_2(s) = \frac{2(e^{\gamma s} - 1)}{(\gamma-b)(e^{\gamma s} - 1) + 2\gamma} \quad (3.6)$$

and $\gamma = \sqrt{b^2 + 2\sigma_\mu^2}$. For details refer to Dahl and Moller (2006). The density function

\(^7\) From the Gompertz model $\mu_x(0) = ye^{bx}$, we have $\mu_{x+t}(t) = \mu_x(0)e^{bt}$ which satisfies Eq.(3.4) with $\sigma_\mu = a = 0$.


\(^9\) It can be shown that if $a \geq \sigma_\mu^2/2$ then the mortality intensity process Eq.(3.4) is strictly positive, see Filipovic (2009).

\(^10\) Although past mortality data shows differences between the evolution of mortality rates for different ages, applying a multi-factor mortality model to the pricing of GLWB requires significant computational resources (refer to Section 4) without adding particular insights to the main results. For this reason we restrict ourselves to a one-factor model.
of the remaining lifetime is then

\[ f_{x+t}(s) = -\frac{d}{ds}sP_{x+t} = -\frac{dC_1(s)}{ds}e^{-C_2(s)\mu_{x+t}(t)} + \frac{dC_2(s)}{ds}sP_{x+t}\mu_{x+t}(t) \]  

(3.7)

where

\[ \frac{dC_1(s)}{ds} = C_1(s)\frac{2a(\gamma - b)}{\sigma^2} \left( \frac{1}{2} - \frac{\gamma e^{\gamma s}}{(\gamma - b)(e^{\gamma s} - 1) + 2\gamma} \right) \]  

(3.8)

and

\[ \frac{dC_2(s)}{ds} = \frac{2\gamma e^{\gamma s}}{(\gamma - b)(e^{\gamma s} - 1) + 2\gamma} \left( 1 - \frac{1}{2}(\gamma - b)C_2(s) \right). \]  

(3.9)

For the case when \( \sigma_{\mu} = 0 \), the survival probability and the density functions are given by

\[ sP_{x+t} = e^{\frac{2}{\sigma^2}\left(\frac{1}{2} - e^{bs}\right)(\mu_{x+t}(t) + \frac{a}{b})} \]  

(3.10)

and

\[ f_{x+t}(s) = sP_{x+t}\left( \frac{a}{b}(e^{bs} - 1) + e^{bs}\mu_{x+t}(t) \right), \]  

(3.11)

respectively. By setting \( a = 0 \) in Eq.(3.10) and Eq.(3.11) we obtain the survival probability and density function of the Gompertz model.

**Remark 1** Luciano and Vigna (2008) study a similar model with \( a = 0 \) and \( \sigma_{\mu} > 0 \). However, such a specification could be problematic since, as it can be shown from Eq.(3.10), we would have

\[ \lim_{s \to \infty} sP_{x+t} = e^{-\frac{2}{\sigma^2}\mu_{x+t}(t)}, \]

that is, the survival probability converges to \( e^{-\frac{2}{\sigma^2}\mu_{x+t}(t)} \) which only approaches zero when \( \sigma_{\mu} = 0 \). It turns out that if we set \( a \neq 0 \) this problem is not present. The mortality intensity \( \mu_{x+t}(t) \) under such a specification has a non zero probability of reaching zero when \( \sigma_{\mu} > 0 \) (regardless of the value of \( b \)). Therefore, assuming \( a \neq 0 \), while \( \sigma_{\mu} > 0 \), the dynamics specified in Eq.(3.4) produces a more satisfactory stochastic model for the mortality intensity process.

### 3.3 Risk-Adjusted Measure

For the purpose of no-arbitrage valuation and hedging, we require the dynamics of the account process \( A(t) \) and the mortality intensity \( \mu_{x+t}(t) \) to be written under a risk-adjusted measure \( Q \). We define \( W_1^Q(t) \) and \( W_2^Q(t) \) as

\[ dW_1^Q(t) = \frac{\mu - r}{\sigma} dt + dW_1(t) \]

\[ dW_2^Q(t) = \lambda \sqrt{\mu_{x+t}(t)} dt + dW_2(t). \]  

(3.12)

By the Girsanov Theorem, see e.g. Bjork (2009), these are standard Brownian motions under the \( Q \) measure with \( \frac{\mu - r}{\sigma} \) and \( \lambda \sqrt{\mu_{x+t}(t)} \) representing the market price of equity.
risk and systematic mortality risk, respectively. We can then write the investment account process and the mortality intensity under \( Q \) as follows:

\[
dA(t) = (r - \alpha g) A(t) dt - G dt + \pi \sigma A(t) dW_1^Q(t) \tag{3.13}
\]

\[
d\mu_{x+t}(t) = (a + (b - \lambda \sigma_{\mu})) \mu_{x+t}(t) dt + \sigma_{\mu} \sqrt{\mu_{x+t}(t)} dW_2^Q(t). \tag{3.14}
\]

By imposing the market price of systematic mortality risk to be \( \lambda \sqrt{\mu_{x+t}(t)} \), the mortality intensity process is still a time homogeneous affine process under \( Q \), as can be seen from Eq.(3.14).\(^{11}\) Thus, this specification preserves tractability of the model when pricing the guarantee under \( Q \). From Eq.(3.13) we observe that the fraction \( \pi \) invested in the equity market only affects the volatility of the investment account process under \( Q \).

![Fig. 3. Survival probabilities and densities of the remaining lifetime of an individual aged 65 with different values of \( \sigma_{\mu} \). Other parameters are as specified in Table 1 with \( \lambda = 0.4 \).](image)

![Fig. 4. Survival probabilities and densities of the remaining lifetime of an individual aged 65 with different values of \( \lambda \). Other parameters are as specified in Table 1. The case \( \lambda = 0 \) corresponds to the survival curve under \( P \).](image)

\(^{11}\)The mortality intensity process might become mean reverting, however, when \( b < \lambda \sigma_{\mu} \).
Using parameters estimated for the stochastic model in Table 1 and allowing volatility of mortality $\sigma_\mu$ to vary, we plot survival probabilities and density functions of the remaining lifetime of an individual aged 65 in the left and the right panel of Fig. 3, respectively. The estimated parameters produce reasonable survival probabilities and density functions of the remaining lifetime. In particular, an increase in $\sigma_\mu$ leads to an improvement in survival probability under the risk-adjusted measure $Q$. Hence, higher volatility of mortality leads not only to higher uncertainty about the timing of death of an individual, but also to an increase in the survival probability. Similar figures are obtained when the market price of systematic mortality risk coefficient $\lambda$ is varying while all other parameters are fixed to the values specified in Table 1, see Fig. 4. Negative values of $\lambda$ indicate a decline in survival probability, while positive values lead to an improvement in survival probability under the risk-adjusted measure $Q$. Thus, under the proposed model specification, a negative value of $\lambda$ is suitable for setting the risk premium for a life insurance portfolio while a positive value of $\lambda$ is adequate for a life annuity business.\textsuperscript{12} Corresponding expected remaining lifetime of an individual as a function of $\sigma_\mu$ and $\lambda$, given $\mathcal{G}_0$, are plotted in the left and right panel of Fig. 5 respectively.

4 Valuation of the GLWB

We begin by generalizing the pricing formulation of guaranteed minimum withdraw benefits (GMWB) in Kolkiewicz and Liu (2012) to the case of the GLWB. There are two approaches to decompose the cashflows involved in the benefit and the premium structure of GMWB, which are shown to be equivalent in Kolkiewicz and Liu (2012). The first approach was suggested by Milevsky and Salisbury (2006) while the second approach was introduced in Aase and Persson (1994) and Persson (1994) under

\textsuperscript{12}If the longevity market is mature enough, $\lambda$ can be instead obtained via calibration to market prices of longevity instruments following the no-arbitrage principle, see for instance Russo et al. (2011).
the “principle of equivalence under $\mathbb{Q}$”. In the following, we present both valuation approaches for the case of GLWB and demonstrate that these are indeed equivalent at any time $t$. We focus on the plain GLWB with assumptions that policyholders exhibit static withdrawal behavior and withdrawals start immediately without any defer period.

4.1 First Approach: Policyholder’s Perspective

Recalling the terminology above, we denote $A(0)$ the initial investment, $g$ the guaranteed withdrawal rate, $\hat{\tau}$ the remaining lifetime of an individual aged $x$ at time 0, $\omega$ the maximum age allowed in the model and $\Pi_{x+t}$ the survival probability of an individual aged $x + t$ at time $t$. From a policyholder’s perspective, income from static withdrawals can be regarded as an immediate life annuity. The no-arbitrage value at time $t$, denoted by $V^P_1(t)$, of an immediate life annuity can be expressed as

$$V^P_1(t) = 1_{\{\hat{\tau} > t\}} g A(0) \int_0^{\omega - x - t} s \Pi_{x+t} e^{-rs} ds,$$  \hspace{1cm} (4.1)

where $0 \leq t \leq \omega - x$ and $1_{\{\hat{\tau} > t\}}$ denotes the indicator function taking value of one if an individual is still alive at time $t$, and zero otherwise.

Let $\tilde{A}(\cdot)$ be the solution to the SDE (3.3) without the condition that $\tilde{A}(\cdot)$ is absorbed at zero. Since the account value cannot be negative and any remaining amount in the account at the time of death of the policyholder is returned to the policyholder’s beneficiary, this cash inflow can be regarded as a call option with payoff $(\tilde{A}(\hat{\tau}))^+$. By the assumption that equity risk and systematic mortality risk are independent, we can write the no-arbitrage value of this call option payoff at time $t$ as

$$V^P_2(t) = 1_{\{\hat{\tau} > t\}} \int_0^{\omega - x - t} f_{x+t}(s) E^Q_t \left( e^{-rs} (\tilde{A}(t + s))^+ \right) ds,$$ \hspace{1cm} (4.2)

where $f_{x+t}(s) = -\frac{d}{ds}(s \Pi_{x+t})$ is the density function of the remaining lifetime of an individual aged $x + t$ at time $t$.

Both, $V^P_1(t)$ and $V^P_2(t)$ are cash inflows while the amount in the investment account $A(t)$ is viewed as a cash outflow to the VA provider. Under the first approach the value of GLWB, denoted by $V^P(t)$, is defined as

$$V^P(t) = V^P_1(t) + V^P_2(t) - 1_{\{\hat{\tau} > t\}} A(t),$$ \hspace{1cm} (4.3)

which can be rewritten as

$$V^P(t) = 1_{\{\hat{\tau} > t\}} \left( \int_0^{\omega - x - t} f_{x+t}(s) \left( \frac{g A(0)}{r} \left( 1 - e^{-rs} \right) + E^Q_t \left( e^{-rs} (\tilde{A}(t + s))^+ \right) \right) ds - A(t) \right),$$ \hspace{1cm} (4.4)

where integration by parts has been applied in order to express Eq.(4.1) in terms of the remaining lifetime density $f_{x+t}(s)$. To make the guarantee fair to both, the
policyholder and the insurer, at time \( t = 0 \) must hold
\[ V^P_1(0) + V^P_2(0) = A(0). \]  
(4.5)

The fair guarantee fee rate, denoted by \( \alpha^*_g \), is defined as the guarantee fee rate that solves Eq.(4.5).

### 4.2 Second Approach: Insurer’s Perspective

Under the second approach the value of GLWB is defined as the expected discounted benefits minus the expected discounted premiums. Let \( \hat{\tau} \) be a random variable defined by
\[ \inf\{ u \geq 0 \mid A(u) = 0 \}, \]
that is, \( \hat{\tau} \) is the time when account value \( A(\cdot) \) is depleted. We can express the expected discounted benefits \( V^I_1(t) \) as
\[ V^I_1(t) = 1_{\{\hat{\tau} > t\}} \int_0^{\infty - x - t} f_{x+t}(s) E_t^Q \left( \int_{t+\hat{\tau}}^{t+s} gA(0)e^{-r(v-t)}1_{\{v > \hat{\tau}\}} dv \right) ds \]
\[ = 1_{\{\hat{\tau} > t\}} \int_0^{\infty - x - t} f_{x+t}(s) \left( \frac{gA(0)}{r} \right) E_t^Q \left( (e^{-r\hat{\tau}} - e^{-rs})^+ \right) ds \]  
(4.6)

and the expected discounted premiums \( V^I_2(t) \) as
\[ V^I_2(t) = 1_{\{\hat{\tau} > t\}} \int_0^{\infty - x - t} f_{x+t}(s) E_t^Q \left( \int_t^{t+(\hat{\tau} \wedge s)} e^{-r(v-t)} \alpha_g A(v) dv \right) ds, \]  
(4.7)

where \( x_1 \wedge x_2 = \min\{x_1, x_2\} \). Under the second approach, the value of GLWB at time \( t \) is defined as
\[ V^I(t) = V^I_1(t) - V^I_2(t), \]  
(4.8)

and the fair guarantee fee rate can be calculated by solving
\[ V^I(0) = 0. \]  
(4.9)

The second approach follows the principle of equivalence under a risk-adjusted measure \( Q \), refer to Aase and Persson (1994) and Persson (1994) for details.

### 4.3 Equivalence of the Two Valuation Approaches

To show that both approaches are equivalent, we consider the following investment strategy. Suppose an individual invests an amount \( A(0) \) in a mutual fund-type account held by an insurance company at time \( t = 0 \). In the future, an individual receives cash flow from the account consisting of, firstly, an amount \( K = k \cdot A(0) \) per unit of time and secondly, an amount \( \beta \cdot A(t) \) per unit of time, which is proportional to the account value. Both amounts are withdrawn continuously until the account is depleted
or individual dies. Finally, any remaining amount in the account will be returned to the individual’s beneficiary. In order to value the entire cash flow including the initial investment, we notice that the insurer provides no financial obligation or guarantee to the above arrangement, that is, the role of the insurer is redundant. Thus, the no-arbitrage value of the above cash flow must be zero at time $t = 0$. That is, we have

$$E_Q^0 \left( -A(0) + \int_0^{\bar{\tau}\wedge \hat{\tau}} e^{-rv}(K + \beta A(v))dv + e^{-r\hat{\tau}}(\tilde{A}(\hat{\tau}))^+ \right) = 0$$

(4.10)

where $k \geq 0$, $\beta \geq 0$ and $A(\tilde{\tau})$ is written as $(\tilde{A}(\tilde{\tau}))^+$.

Using the fact that $\int_0^{\bar{\tau}\wedge \hat{\tau}} = \int_0^{\hat{\tau}}1_{\{\tau > \hat{\tau}\}}$ and rearranging the terms in Eq.(4.10), we obtain

$$E_Q^0 \left( -A(0) + \int_0^{\hat{\tau}} K e^{-rv}dv + e^{-r\hat{\tau}}(\tilde{A}(\hat{\tau}))^+ \right) = E_Q^0 \left( \int_0^{\hat{\tau}} e^{-rv}K1_{\{\tau > \hat{\tau}\}}dv - \int_0^{\bar{\tau}\wedge \hat{\tau}} e^{-rv}\beta A(v)dv \right)$$

(4.11)

The L.H.S. and R.H.S. of Eq.(4.11) are $V^P(0)$ and $V^I(0)$, respectively. Note that the parameter $k$ can be interpreted as the GLWB guaranteed withdrawal rate $g$, whereas $\beta$ is the guarantee fee rate corresponding to $\alpha$. By imposing the principle of equivalence under $Q$ we note that the R.H.S. of Eq.(4.11) equals zero, which implies that the L.H.S. of Eq.(4.11) must be equal to zero as well. Since the argument does not rely on whether the remaining lifetime $\tilde{\tau}$ is stochastic or deterministic, it is evident that the equivalence holds for the valuation of GMWB where $\tilde{\tau} = T = 1/k$ is deterministic, assuming static withdrawal behavior. Table 2 reports in the last two columns the fair guarantee fee rate $\alpha^*_g$ computed using the two equivalent approaches summarized above. The result suggests that both approaches lead to the same $\alpha^*_g$, subject to simulation error with finite sample size.

Table 2

<table>
<thead>
<tr>
<th>Case</th>
<th>$r$</th>
<th>$\sigma$</th>
<th>$g$</th>
<th>$\sigma_\mu$</th>
<th>$\lambda$</th>
<th>Fee (1st App.)</th>
<th>Fee (2nd App.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>4%</td>
<td>20%</td>
<td>5.0%</td>
<td>0.04</td>
<td>0.2</td>
<td>0.48%</td>
<td>0.48%</td>
</tr>
<tr>
<td>2.</td>
<td>3%</td>
<td>25%</td>
<td>4.5%</td>
<td>0.02</td>
<td>0.0</td>
<td>0.45%</td>
<td>0.45%</td>
</tr>
<tr>
<td>3.</td>
<td>5%</td>
<td>15%</td>
<td>5.5%</td>
<td>0.05</td>
<td>0.8</td>
<td>0.50%</td>
<td>0.49%</td>
</tr>
<tr>
<td>4.</td>
<td>4%</td>
<td>20%</td>
<td>6.0%</td>
<td>0.00</td>
<td>0.0</td>
<td>0.71%</td>
<td>0.71%</td>
</tr>
</tbody>
</table>

One can easily generalize Eq.(4.10) to any time $t \geq 0$. Both approaches defined through Eq.(4.5) and Eq.(4.9) can be applied to value GLWB at any time $t \geq 0$.

While the first approach is computationally more efficient, the second approach highlights the theoretical result that the market reserve of a payment process is defined as the expected discounted benefit minus the expected discounted premium under a risk-adjusted measure, see Dahl and Moller (2006).

**Remark 2** For some special cases the equivalence, formulated in Eq.(4.11), can be verified analytically. If $A(t)$ follows the dynamics in SDE (3.3) and
(1) $k = 0$ and $\beta \geq 0$, Eq.(4.11) becomes

$$A(0) = E^Q_0 \left( e^{-r\tau} A(\tau) + \int_0^\tau e^{-rs} \beta A(s) ds \right),$$

where the guarantee fee rate $\beta$ can be interpreted as a continuous dividend yield, see e.g., Bjork (2009). In particular, when $\beta = 0$, it states that the discounted asset is a $Q$-martingale.

(2) $k > 0$, $\beta = 0$ and $\sigma = 0$, we have

$$dA(t) = (rA(t) - kA(0))dt$$

which has the solution

$$A(t) = A(0)e^{rt} + \frac{kA(0)}{r} \left( 1 - e^{rt} \right).$$

The account $A(t)$ can be interpreted as a money market account where a constant amount is withdrawn continuously. Solving $A(u) = 0$ we obtain $u = \frac{1}{r} \ln \frac{k}{e^{rt} - r}$. We can verify Eq.(4.11) by noticing that

$$E^Q_0 \left( \int_0^{u_x} e^{-rv} Kdv + e^{-r\tilde{\tau}} (\tilde{A}(\tilde{\tau}))^+ \right) = \int_0^{\omega - x} f_x(s) \left( \int_0^{u_x} e^{-rv} Kdv + e^{-rs} (\tilde{A}(s))^+ \right) ds = A(0)$$

where the second equality is obtained by considering two cases $s \geq u$ and $s < u$ separately.

5 Sensitivity Analysis

As discussed in Section 1, risk assessment is important for analyzing the underlying risks of an insurance product. Given that the underlying guarantee is a long term contract, the effect of parameter risk on pricing is expected to be significant. We use sensitivity analysis as a risk assessment to quantify the impact of parameter risk on GLWB pricing.

We investigate the relationship between the fair guaranteed fee rate $\alpha_g^*$ and important financial and demographic factors, which include the age of the insured, the equity exposure $\pi$, the interest rate level $r$, the volatility of mortality $\sigma_\mu$ and the market price of systematic mortality risk coefficient $\lambda$. Table 3 summarizes parameters for the base case, which will be used to study the effect of different risks on the fair guaranteed fee rate $\alpha_g^*$.

Table 3

<table>
<thead>
<tr>
<th>$r$</th>
<th>$A(0)$</th>
<th>$\sigma$</th>
<th>$\pi$</th>
<th>$\gamma$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\sigma_\mu$</th>
<th>$\lambda$</th>
<th>$\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4%</td>
<td>100</td>
<td>25%</td>
<td>0.7</td>
<td>5%</td>
<td>0.001</td>
<td>0.087</td>
<td>0.021</td>
<td>0.4</td>
<td>110</td>
</tr>
</tbody>
</table>
5.1 Sensitivity to Mortality Risk

Fig. 6 shows the impact of volatility of mortality $\sigma_{\mu}$ on the fair guaranteed fee rate $\alpha^*_g$ for policyholders aged 65 (left panel) and 75 (right panel) with different values of the guaranteed withdrawal rate $g$. One observes that $\alpha^*_g$ is approximately an exponential function of $\sigma_{\mu}$, which is consistent with the fact that the expected remaining lifetime increases exponentially with respect to $\sigma_{\mu}$, see the left panel of Fig. 5.

The values of the guaranteed withdrawal rate $g$ have a significant effect on the fair guaranteed fee rate. When $g$ increases from 5% to 5.5% (a relative increase of 10%), $\alpha^*_g$ increases from 0.5% to 0.8% (a relative increase of 60%) for the case when $\sigma_{\mu} = 0.021$. This highlights how insurance providers can reduce benefits of the guarantee by focusing on $g$ rather than $\alpha^*_g$, since changing the guaranteed withdrawal rate appears to have smaller impact on the benefits than changing the fair guaranteed fee rate from a policyholder’s perspective. Furthermore, the fair guaranteed fee rates are significantly lower for a 75 year old compared to a 65 year old, assuming that they have the same amount of retirement savings. This follows from the fact that a 75 year old has a lower probability of exhausting the retirement savings in the account before he or she dies under the condition that the annual withdrawal amount is limited. Even if the account is depleted before the policyholder dies, the period of the income guaranteed by the provider for a 75 year old is expected to be shorter than for a 65 year old.

The effect of the market price of systematic mortality risk coefficient $\lambda$ on the fair guaranteed fee rate is similar to the effect of the volatility of mortality on $\alpha^*_g$, see Fig. 7. Explanation for this result can be carried over from the case of the volatility of mortality. Fig. 5, 6 and 7 show that the effect of varying parameters related to the systematic mortality risk on pricing can be reliably indicated using the expected remaining lifetime as a function of the underlying parameters.

Fig. 6. Sensitivity of the fair guaranteed fee rate $\alpha^*_g$ with respect to volatility of mortality $\sigma_{\mu}$ for policyholders aged 65 (left panel) and 75 (right panel).
Fig. 7. Sensitivity of the fair guaranteed fee rate $\alpha^*_g$ with respect to market price of systematic mortality coefficient $\lambda$ for policyholders aged 65 (left panel) and 75 (right panel).

5.2 Sensitivity to Financial Risk

Sensitivity of the fair guaranteed fee rate with respect to the equity exposure $\pi$, measured by the volatility of the investment account $\pi \cdot \sigma$, is shown in Fig. 8 where $\sigma$ is set equal 25% and $\pi \in \{0, 0.3, 0.5, 0.7, 1\}$. The fair guaranteed fee rate is very sensitive to the equity exposure as $\alpha^*_g$ increases rapidly when $\pi$ (and hence, $\pi \cdot \sigma$) increases. Positive relationship between $\alpha^*_g$ and $\pi \cdot \sigma$ is consistent with financial theory where options are more expensive when volatility of underlying is high. When $\pi$ is close to zero, $\alpha^*_g$ approaches zero as well for $g \leq 5\%$. The intuition behind this is that there is essentially no liability for the guarantee provider since the guaranteed withdrawal rate is too low for the case when $\pi \cdot \sigma = 0$. Numerical experiments indicate that it takes more than 35 years for the account value to drop to zero when $\pi \cdot \sigma = 0$ with $g = 5\%$, and only few policyholders are still alive by that time. In this situation the income guaranteed by the insurer will never be realized. Similarly to the case of the systematic mortality risk, the fair guaranteed fee rate is sensitive to $g$, as well as to the age of the policyholder.

The relationship between the interest rate level $r$ and $\alpha^*_g$ is negative, see Fig. 9, in contrast to the case of equity exposure and parameters related to mortality risk. This can be explained by considering Eq.(4.8) and Eq.(4.9) where the fair guaranteed fee rate is defined as the fee rate that makes the expected discounted benefit equal to the expected discounted premium. As $r$ increases, the present value of the benefit and the premium decreases. However, since the benefit is further away in the future than the premium, the present value of the benefit will drop faster than the present value of the premium. As a result, a lower value of $\alpha^*_g$ is required to make the expected discounted benefit equal to the expected discounted premium, where $\alpha^*_g$ has a larger impact on the expected discounted premium component.

Another interesting result arising from Fig. 9 is that the fair guaranteed fee rate is very sensitive to low interest rates. For instance, when $r$ drops from 2% to 1% for a 65 year old policyholder while $g$ is set equal 5.5%, $\alpha^*_g$ increases approximately from 3% to 7.5%. This result indicates a challenging situation for GLWB providers who
issue the guarantee when interest rates are high, but subsequently decline to a very low level where they remain for a prolonged period of time. Fair guaranteed fee rates might be substantially undervalued in this circumstances. \(^{13}\)

Fig. 8. Sensitivity of the fair guaranteed fee rate \(\alpha_g^*\) with respect to equity exposure \(\pi\), represented by volatility of the investment account \(\pi \cdot \sigma\), for policyholders aged 65 (left panel) and 75 (right panel).

Fig. 9. Sensitivity of the fair guaranteed fee rate \(\alpha_g^*\) with respect to interest rate level \(r\) for policyholders aged 65 (left panel) and 75 (right panel).

5.3 Sensitivity to Parameter Risk

A significant issue related to the long term nature of GLWB is that misspecified parameters at the inception of the contract can have a significant impact on pricing, which in turn affects the realized profit and loss (P&L) for the guarantee providers. Even if the proposed modeling framework effectively captures the qualitative features

\(^{13}\) The impact of market variables volatility on guarantees in variable annuities, including low interest rate environment in the US starting with the global financial crisis in 2007-2008, is discussed in “American VA providers de-risk to combat market volatility”, January 2012, Risk Magazine.
of the underlying financial and demographic variables, estimation of model parameters is a challenging task. The challenge arises partly due to the fact that markets for the long term contracts are relatively illiquid, and hence no reliable market prices are available for parameter calibration. However, one could rely on historical data, perhaps combined with expert judgements, when estimating parameters and pricing a long term contract. Given the long term nature of the guarantee, perfect hedging is economically unviable when taking transaction costs and liquidity into account. In these circumstances the insurance provider must accept a certain level of parameter risk. Sensitivity analysis of the relative impact of these risks inherent in parameter (mis)specification on pricing of the guarantee is an important step towards understanding the risks undertaken by the guarantee providers.

In the following we study the impact of different model parameters on pricing of GLWB. As shown in Fig. 9, the fair guaranteed fee rate is extremely sensitive to low interest rates. The level of interest rates is important when pricing GLWB, especially in a prolonged period of low interest rate environment. Fig. 6 and Fig. 8 show how equity exposure \( \pi \), and hence volatility of the investment account \( \pi \cdot \sigma \), have a larger impact on pricing of GLWB than the volatility of mortality \( \sigma_\mu \). However, the parameter risk of \( \sigma_\mu \) is non-negligible, which is illustrated by considering the following scenario. Suppose that the volatility of mortality \( \sigma_\mu = 0.02 \) is fixed (all other parameters are as specified in Table 3) and the guarantee provider estimates fund’s volatility to be \( \sigma = 20\% \) at the time when the guarantee is issued. However, it turns out in the future that the “true” parameter corresponds to \( \sigma = 25\% \). In this scenario the guarantee provider has misspecified \( \sigma \) by 5\%. This leads to an underestimation of the guaranteed fee rate \( \alpha_g^* \) by approx. \( (0.6 - 0.4)\% = 0.2\% \), see Fig. 8 (for the case \( g = 5\% \)). On the other hand, the same magnitude of mis(pricing) of \( \alpha_g^* \) (0.2\%) can be observed when \( \sigma = 25\% \) is fixed while the volatility of mortality \( \sigma_\mu \) is misspecified by approximately \( (0.04 - 0.02) = 0.02 \) (if the original estimation of \( \sigma_\mu \) were 0.02 at the inception of the guarantee while the “true” parameter value turns out to be \( \sigma_\mu = 0.04 \) in the future), see Fig. 6 (for the case \( g = 5\% \)). If volatility of mortality changes from \( \sigma_\mu = 0.02 \) to \( \sigma_\mu = 0.04 \), the expected remaining lifetime would increase by approximately \( (21 - 18) = 3 \) years, see Fig. 5. An underestimation of three years of expected remaining lifetime for a period of 30 to 40 years\(^{14}\) is possible and the risk of misspecification of the volatility of mortality cannot be ignored.

6 Profit and Loss Analysis

Although sensitivity analysis provides us with a quantitative measure for the degree of impact of parameter risk on pricing, profit and loss (P&L) analysis can provide another perspective especially when hedging is not available or difficult to carry out. Hari et al. (2008) use P&L analysis to assess systematic mortality risk in pension annuities.

In order to better assess the underlying risks involved in GLWB, we simulate profit

\(^{14}\) This corresponds approximately to the duration of a VA portfolio with GLWB.
and loss (P&L) distributions for different scenarios where no hedging is performed and liabilities are funded solely by the guaranteed fee charged by the insurer. We assume that the VA portfolio consists of 1000 policyholders all aged 65 who have elected GLWB as the only guarantee to protect their investment. Each individual invests $100 and chooses the same equity exposure $\pi$ for their retirement savings investment. A fair guaranteed fee is charged according to the valuation result in Section 4.

The P&L for each individual is obtained by simulating the account process $A(\cdot)$. If policyholder dies before the account is depleted, the profit for the insurer is determined by the received fee, which grows at an interest rate $r$. On the other hand, if policyholder dies after the account value is depleted, the fee charged by the insurer will be used to fund incurred liabilities. We simulate the mortality intensity process $\mu_{65+}$ for each scenario and apply the inverse transform method (Brigo and Mercurio (2006)) to obtain 1,000 independent remaining lifetimes. The surplus/shortfall is aggregated across all individuals, divided by the total investment ($1,000 \cdot $100 = $100,000), and discounted to obtain the discounted P&L of the portfolio per dollar received.$^{15}$ The described procedure constitutes one sample. To obtain summary statistics for the discounted P&L distribution, 100,000 simulations are performed.

Summary statistics for the discounted P&L distributions include the fair guaranteed fee rate $\alpha^*_g$, the mean P&L for the insurer; its standard deviation; the coefficient of variation (C.V.) defined as the ratio of standard deviation over the mean P&L; the Value-at-Risk (VaR) defined as the $q$%-quantile of the P&L distribution; and the Expected Shortfall (ES) determined as the expected loss of the portfolio given that the loss occurs at or below the $q$%-quantile. The confidence level for the VaR and the ES corresponds to 99.5%. Parameter values used in the simulations are as specified in Table 3 except for the guaranteed withdrawal rate $g$ which is set equal to 6%.$^{16}$

### 6.1 P&L with or without systematic mortality risk

We first consider interaction between systematic mortality risk and equity exposure $\pi$, and its effect on the P&L. Table 4 reports summary statistics for the discounted P&L distributions with systematic mortality risk (with s.m.r) or without systematic mortality risk (no s.m.r.). We first consider the situation when $\pi = 0$, that is, the retirement savings is entirely invested in the money market account with deterministic growth. Since portfolio is large enough, the unsystematic mortality risk is diversified away. When $\pi = 0$ and there is no s.m.r., the fee charged by the insurer is able to almost exactly (on average) offset the liability.

---

$^{15}$Since the time of death of a policyholder is random and the maturity of the contract is not fixed, using discounted P&L rather than P&L allows for a more convenient comparison and interpretation of the results at time $t = 0$.

$^{16}$When equity exposure $\pi = 0$, the guarantee is too generous for the insurer. This leads to positive VaR and ES even for a relatively high confidence level 99.5%. For the purpose of convenience of interpretation of the results, we increase $g$ from 5% to 6%, which leads to negative VaR and ES across all equity exposures.
Once systematic mortality risk is introduced, the average return, as well as the risk, determined by the standard deviation, of selling the guarantee, increases, which is consistent with a standard risk-return tradeoff argument. The increase in the average return can be partly attributed to the assumption that $\lambda = 0.4$, that is, a positive risk premium associated with the systematic mortality risk, see Fig. 4. C.V. takes its highest (lowest) value of 1.984 (0.727) in a presence of (without) systematic mortality risk and no equity exposure. Generally, distributions with $C.V. > 1$ ($C.V. < 1$) are considered to be high- (low-) variance. For the case of no systematic mortality risk we observe that C.V. increases with increasing equity exposure. This indicates that for high values of $\pi$ the risk determined by the standard deviation increases faster than the return, compared to the situation when $\pi$ is low. VaR and ES increase (in absolute terms) when we move from no s.m.r. to s.m.r., and when equity exposure level $\pi$ increases. In addition, one observes that the effect of systematic mortality risk becomes less pronounced when $\pi$ increases. This shows that if policyholders choose low exposure to equity risk, the guarantee provider should be more careful about systematic mortality risk, as it becomes a larger contributor to the overall risk underlying the guarantee.

### Table 4

The effect of systematic mortality risk with respect to different levels of equity exposure $\pi$. Summary statistics are computed using discounted P&L distribution per dollar received.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha^*_g$</th>
<th>Mean</th>
<th>Std.dev.</th>
<th>C.V.</th>
<th>VaR$_{0.995}$</th>
<th>ES$_{0.995}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\pi = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>no s.m.r.</td>
<td>0.09%</td>
<td>0.0011</td>
<td>0.0008</td>
<td>0.727</td>
<td>-0.0010</td>
<td>-0.0013</td>
</tr>
<tr>
<td>with s.m.r.</td>
<td>0.23%</td>
<td>0.0062</td>
<td>0.0123</td>
<td>1.984</td>
<td>-0.0470</td>
<td>-0.0556</td>
</tr>
<tr>
<td>$\pi = 0.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>no s.m.r.</td>
<td>0.32%</td>
<td>0.0267</td>
<td>0.0243</td>
<td>0.909</td>
<td>-0.0807</td>
<td>-0.0995</td>
</tr>
<tr>
<td>with s.m.r.</td>
<td>0.51%</td>
<td>0.0427</td>
<td>0.0363</td>
<td>0.850</td>
<td>-0.1147</td>
<td>-0.1444</td>
</tr>
<tr>
<td>$\pi = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>no s.m.r.</td>
<td>0.62%</td>
<td>0.0590</td>
<td>0.0624</td>
<td>1.057</td>
<td>-0.1595</td>
<td>-0.1863</td>
</tr>
<tr>
<td>with s.m.r.</td>
<td>0.85%</td>
<td>0.0807</td>
<td>0.0838</td>
<td>1.038</td>
<td>-0.1988</td>
<td>-0.2369</td>
</tr>
<tr>
<td>$\pi = 0.7$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>no s.m.r.</td>
<td>0.97%</td>
<td>0.1049</td>
<td>0.1318</td>
<td>1.256</td>
<td>-0.2333</td>
<td>-0.2619</td>
</tr>
<tr>
<td>with s.m.r.</td>
<td>1.25%</td>
<td>0.1362</td>
<td>0.1717</td>
<td>1.261</td>
<td>-0.2718</td>
<td>-0.3148</td>
</tr>
<tr>
<td>$\pi = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>no s.m.r.</td>
<td>1.60%</td>
<td>0.2136</td>
<td>0.3579</td>
<td>1.675</td>
<td>-0.3236</td>
<td>-0.3532</td>
</tr>
<tr>
<td>with s.m.r.</td>
<td>1.88%</td>
<td>0.2520</td>
<td>0.4599</td>
<td>1.792</td>
<td>-0.3628</td>
<td>-0.4047</td>
</tr>
</tbody>
</table>

### 6.2 Parameter Risk

As specified in Section 5.3, parameter risk refers to the situation when parameters from the model are not specified/estimated correctly by the guarantee provider. Table 5 shows distributional statistics for the discounted P&L when different parameters are assumed to be misspecified. These include financial risk parameters such as interest rate $r$ and fund volatility $\sigma$, as well as demographic risk parameters such as volatility of mortality $\sigma_\mu$ and systematic mortality risk coefficient $\lambda$. The first column of Table 5 reports parameters $r$, $\sigma$, $\sigma_\mu$ and $\lambda$ (mis)specified by the guarantee provider at the initiation of the contract, whereas the “true” parameters corresponding to $r = 4\%$, $\sigma = 25\%$, $\sigma_\mu = 0.021$ and $\lambda = 0.4$ are indicated in bold letters. The second column represents the fair guaranteed fee rate calculated using the misspec-
ified parameters, with other parameters set to be “true”. For instance, if \( \sigma \) is set to be 20% by the guarantee provider, the fair guaranteed fee rate is underestimated by 1.25% – 0.97% = 0.28%.

Table 5
Parameter risk: distributional statistics for the discounted P&L per dollar received. The "true" parameters corresponding to \( r = 4\% \), \( \sigma = 25\% \), \( \sigma_\mu = 0.021 \) and \( \lambda = 0.4 \) and are indicated in bold letters.

<table>
<thead>
<tr>
<th>Case</th>
<th>( \alpha_g )</th>
<th>Mean</th>
<th>Std.dev.</th>
<th>( \text{VaR}_{0.995} )</th>
<th>( \text{ES}_{0.995} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>5.70%</td>
<td>0.3619</td>
<td>0.3461</td>
<td>-0.2210</td>
<td>-0.2636</td>
</tr>
<tr>
<td>3%</td>
<td>2.40%</td>
<td>0.2344</td>
<td>0.2511</td>
<td>-0.2578</td>
<td>-0.3012</td>
</tr>
<tr>
<td>4%</td>
<td>1.25%</td>
<td>0.1362</td>
<td>0.1717</td>
<td>-0.2718</td>
<td>-0.3148</td>
</tr>
<tr>
<td>5%</td>
<td>0.70%</td>
<td>0.0731</td>
<td>0.1214</td>
<td>-0.2855</td>
<td>-0.3264</td>
</tr>
<tr>
<td>6%</td>
<td>0.39%</td>
<td>0.0326</td>
<td>0.0901</td>
<td>-0.2872</td>
<td>-0.3280</td>
</tr>
<tr>
<td>7%</td>
<td>0.20%</td>
<td>0.0055</td>
<td>0.0718</td>
<td>-0.2882</td>
<td>-0.3299</td>
</tr>
<tr>
<td>10%</td>
<td>0.48%</td>
<td>0.0446</td>
<td>0.0995</td>
<td>-0.2882</td>
<td>-0.3292</td>
</tr>
<tr>
<td>15%</td>
<td>0.70%</td>
<td>0.0731</td>
<td>0.1214</td>
<td>-0.2855</td>
<td>-0.3264</td>
</tr>
<tr>
<td>20%</td>
<td>0.97%</td>
<td>0.1058</td>
<td>0.1475</td>
<td>-0.2764</td>
<td>-0.3175</td>
</tr>
<tr>
<td>25%</td>
<td>1.25%</td>
<td>0.1362</td>
<td>0.1717</td>
<td>-0.2718</td>
<td>-0.3148</td>
</tr>
<tr>
<td>30%</td>
<td>1.50%</td>
<td>0.1607</td>
<td>0.1905</td>
<td>-0.2718</td>
<td>-0.3160</td>
</tr>
<tr>
<td>35%</td>
<td>1.90%</td>
<td>0.1962</td>
<td>0.2198</td>
<td>-0.2633</td>
<td>-0.3100</td>
</tr>
<tr>
<td></td>
<td>( \sigma_\mu )</td>
<td>(\sigma_\mu)</td>
<td>(\sigma_\mu)</td>
<td>(\sigma_\mu)</td>
<td>(\sigma_\mu)</td>
</tr>
<tr>
<td>0.005</td>
<td>1.00%</td>
<td>0.1079</td>
<td>0.1487</td>
<td>-0.2816</td>
<td>-0.3221</td>
</tr>
<tr>
<td>0.010</td>
<td>1.08%</td>
<td>0.1175</td>
<td>0.1564</td>
<td>-0.2733</td>
<td>-0.3152</td>
</tr>
<tr>
<td>0.015</td>
<td>1.15%</td>
<td>0.1250</td>
<td>0.1633</td>
<td>-0.2751</td>
<td>-0.3212</td>
</tr>
<tr>
<td>0.021</td>
<td>1.25%</td>
<td>0.1362</td>
<td>0.1717</td>
<td>-0.2718</td>
<td>-0.3148</td>
</tr>
<tr>
<td>0.025</td>
<td>1.36%</td>
<td>0.1472</td>
<td>0.1790</td>
<td>-0.2731</td>
<td>-0.3153</td>
</tr>
<tr>
<td>0.030</td>
<td>1.54%</td>
<td>0.1645</td>
<td>0.1939</td>
<td>-0.2699</td>
<td>-0.3131</td>
</tr>
<tr>
<td></td>
<td>( \lambda )</td>
<td>(\lambda)</td>
<td>(\lambda)</td>
<td>(\lambda)</td>
<td>(\lambda)</td>
</tr>
<tr>
<td>0.1</td>
<td>1.14%</td>
<td>0.1244</td>
<td>0.1619</td>
<td>-0.2743</td>
<td>-0.3186</td>
</tr>
<tr>
<td>0.2</td>
<td>1.16%</td>
<td>0.1263</td>
<td>0.1637</td>
<td>-0.2772</td>
<td>-0.3162</td>
</tr>
<tr>
<td>0.3</td>
<td>1.22%</td>
<td>0.1321</td>
<td>0.1684</td>
<td>-0.2759</td>
<td>-0.3188</td>
</tr>
<tr>
<td>0.4</td>
<td>1.25%</td>
<td>0.1362</td>
<td>0.1717</td>
<td>-0.2718</td>
<td>-0.3148</td>
</tr>
<tr>
<td>0.5</td>
<td>1.30%</td>
<td>0.1412</td>
<td>0.1765</td>
<td>-0.2744</td>
<td>-0.3155</td>
</tr>
<tr>
<td>0.6</td>
<td>1.35%</td>
<td>0.1458</td>
<td>0.1785</td>
<td>-0.2686</td>
<td>-0.3129</td>
</tr>
</tbody>
</table>

The mean P&L depends on the guaranteed fee rate. Overcharging or undercharging the guaranteed fee leads to an increase or decrease in the mean P&L, respectively. In contrast to the mean P&L, risk measures such as VaR and ES are relatively insensitive to the fair guaranteed fee rate. Standard deviation becomes larger when the guaranteed fee rate increases. This is due to the fact that the P&L distribution is affected by two factors: probability of occurrence of different scenarios and the cash flow involved in each scenario. A miscalculation of the guaranteed fee rate has no impact on the probability of occurrence of different scenarios, but does affect the cash flow. VaR and ES as the tail risk measures are mostly concerned with scenarios represented in Fig. 1 where policyholder dies after the account value is depleted and there is liability for the insurer. As the guaranteed fee rate has larger impact on premiums than on liabilities\(^{17}\), and for the worst 0.5% scenarios liabilities are much larger than premiums, overcharging or undercharging the guaranteed fee has only a small impact.

\(^{17}\)Premiums and liabilities are from an insurer’s prospective. Premiums (or guaranteed fees) depend on the guaranteed fee rate directly while liabilities implicitly depend on the fee rate as the investment account will decrease (increase) slightly faster when the fee rate is higher (lower).
on the low tail of the P&L distribution. As a result, VaR and ES are relatively robust to miss-specification of the guaranteed fee rate.

Standard deviation is a measure of dispersion of the data around its mean. Scenarios represented in Fig. 2, when there is liability for the insurer, are typical when considering standard derivation of the P&L, since most samples around the mean have larger premiums than liabilities. To explain why standard deviation becomes larger when the guaranteed fee rate increases, we consider two scenarios, assuming no liabilities. The difference in profits for these two scenarios is calculated as $\alpha_g \int_0^\tau (A_1(t) - A_2(t)) \, dt$, which indicates how “apart” these two samples are when the guaranteed fee rate is $\alpha_g$. Now suppose that the guaranteed fee rate is $\hat{\alpha}_g$ such that $\hat{\alpha}_g \neq \alpha_g$. For the same two scenarios the difference of profits becomes $\hat{\alpha}_g \int_0^\tau (\hat{A}_1(t) - \hat{A}_2(t)) \, dt$. Since the integrand is the difference of two account values and the value of the guaranteed fee rate has only a small effect on the dynamics of $A(t)$, it must approximately hold that $\int_0^\tau (A_1(t) - A_2(t)) \, dt \approx \int_0^\tau (\hat{A}_1(t) - \hat{A}_2(t)) \, dt$. As a result, the samples are more “apart” when $\hat{\alpha}_g > \alpha_g$.

6.3 Model Risk

We assess model risk by considering scenarios where the guarantee provider assumes deterministic mortality model, ignoring systematic mortality risk. To implement deterministic mortality model we set $a = 0$ and $\sigma_\mu = 0$ in Eq.(3.4) and estimate parameter $b$ as discussed in Section 3.2; it corresponds to $b = 0.106$. Table 6 reports the results for the P&L statistics, comparing stochastic and deterministic mortality models. One would expect that for small equity exposures the longevity effect is more pronounced, leading to a larger $\alpha_g^*$ for stochastic mortality model compared to deterministic model. Indeed, from the table one observes that when equity exposure $\pi \leq 0.7$, deterministic mortality leads to an underestimation of the fair guaranteed fee rate $\alpha_g^*$, which results in a decrease of the average return; and the tail risk measures VaR and ES worsen. However, for a relatively large equity exposures, say $\pi > 0.5$, there is essentially no difference between deterministic and stochastic mortality models in terms of the mean return, the VaR and the ES, as well as the resulting fair guaranteed fee rate $\alpha_g^*$. This reflects the fact that for large $\pi$, equity risk dominates systematic mortality risk.

7 Static Hedging of Systematic Mortality Risk

Several longevity-linked securities have been proposed in the literature as hedging instruments for systematic mortality risk. Here we consider the S-forward, or ‘survivor’ forward, which has been developed by LLMA (2010). It is a cash settled contract

\footnote{Note that samples in the P&L distribution come from a portfolio with 1000 policyholders and hence, the death time $\tau$ appearing in the upper limit of integration is representative only.}
Table 6
Model risk assuming stochastic vs. deterministic mortality model: distributional statistics for the discounted P&L per dollar received.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha^\pi_g$</th>
<th>Mean</th>
<th>Std.dev.</th>
<th>VaR$_{0.995}$</th>
<th>ES$_{0.995}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0.23%</td>
<td>0.0062</td>
<td>0.0123</td>
<td>-0.0470</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.14%</td>
<td>0.0001</td>
<td>0.0114</td>
<td>-0.0501</td>
</tr>
<tr>
<td>stochastic</td>
<td>0.50%</td>
<td>0.0427</td>
<td>0.0363</td>
<td>-0.1147</td>
<td>-0.1444</td>
</tr>
<tr>
<td>deterministic</td>
<td>0.44%</td>
<td>0.0371</td>
<td>0.0345</td>
<td>-0.1166</td>
<td>-0.1507</td>
</tr>
<tr>
<td>$\pi = 0.3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>stochastic</td>
<td>0.85%</td>
<td>0.0807</td>
<td>0.0838</td>
<td>-0.1988</td>
<td>-0.2369</td>
</tr>
<tr>
<td>deterministic</td>
<td>0.81%</td>
<td>0.0772</td>
<td>0.0822</td>
<td>-0.2026</td>
<td>-0.2408</td>
</tr>
<tr>
<td>stochastic</td>
<td>1.25%</td>
<td>0.1362</td>
<td>0.1717</td>
<td>-0.2718</td>
<td>-0.3148</td>
</tr>
<tr>
<td>deterministic</td>
<td>1.23%</td>
<td>0.1326</td>
<td>0.1685</td>
<td>-0.2731</td>
<td>-0.3174</td>
</tr>
<tr>
<td>$\pi = 0.5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>stochastic</td>
<td>1.88%</td>
<td>0.2520</td>
<td>0.4599</td>
<td>-0.3628</td>
<td>-0.4047</td>
</tr>
<tr>
<td>deterministic</td>
<td>1.90%</td>
<td>0.2534</td>
<td>0.4478</td>
<td>-0.3684</td>
<td>-0.4117</td>
</tr>
</tbody>
</table>

linked to survival rates of a given population cohort. The S-forward is the basic building block for longevity (survivor) swaps, see Dowd (2003), which are used by pension funds and insurance companies to hedge their longevity risk. Longevity swaps can be regarded as a stream of S-forwards with different maturity dates. The S-forward is an agreement between two counterparties to exchange at maturity an amount equal to the realized survival rate of a given population cohort, with that of a fixed survival rate agreed at the inception of the contract.

One of the advantages of using S-forwards as our choice of hedging strategy is that there is no initial capital requirement at the inception of the contract and cash flows occur only at maturity. This is in line with our P&L analysis. Another benefit of using S-forward is that the underlying of the contract is the survival probability which has a closed form analytical expression under the proposed continuous time mortality modelling framework. Since dynamic hedging requires liquid longevity-linked instruments and the longevity market is illiquid, the choice of using static hedging is more realistic.

An S-forward is a swap with only one payment at maturity $T$ where the fixed leg pays an amount equal to

$$N \cdot E_0^Q \left( e^{-\int_0^T \mu_{x+s}(s) ds} \right) \quad (7.1)$$

while the floating leg pays an amount

$$N \cdot e^{-\int_0^T \mu_{x+s}(s) ds}, \quad (7.2)$$

with $N$ denoting the notional amount of the contract. The fixed leg is determined under the risk-adjusted measure $Q$ which is discussed in Section 3.3. Given that there is a positive risk premium for systematic mortality risk, mortality risk hedger who pays fixed leg and receives floating leg bears implicit cost for entering the S-forward.

Since maturity of the GLWB is not fixed, our numerical example assumes $T = 20$, which approximately corresponds to the expected remaining lifetime of a 65 year old
(Fig. 5). To determine an appropriate notional amount $N$ for the contract, we notice that $N$ should depend on the total investment $I$ (which corresponds to $100 for each of 1000 policyholders: $I = 1000 \times 100 = 100,000$) of a VA portfolio. Assuming $N$ to be a linear function of $I$ ($N = \theta \cdot I$), we found that $\theta \approx 0.35$ provides the most effective hedge in a sense of the VaR and the ES reduction (in absolute terms).

Table 7
Static hedging of systematic mortality risk via S-forward: distributional statistics for the discounted P&L per dollar received.

<table>
<thead>
<tr>
<th>Case</th>
<th>Mean</th>
<th>Std.dev.</th>
<th>VaR$_{0.995}$</th>
<th>ES$_{0.995}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No hedge</td>
<td>0.0062</td>
<td>0.0123</td>
<td>-0.0470</td>
<td>-0.0556</td>
</tr>
<tr>
<td>Static hedge</td>
<td>0.0016</td>
<td>0.0111</td>
<td>-0.0283</td>
<td>-0.0313</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\pi = 0.3$</td>
<td></td>
</tr>
<tr>
<td>No hedge</td>
<td>0.0427</td>
<td>0.0363</td>
<td>-0.1147</td>
<td>-0.1444</td>
</tr>
<tr>
<td>Static hedge</td>
<td>0.0382</td>
<td>0.0396</td>
<td>-0.0995</td>
<td>-0.1251</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\pi = 0.5$</td>
<td></td>
</tr>
<tr>
<td>No hedge</td>
<td>0.0807</td>
<td>0.0838</td>
<td>-0.1988</td>
<td>-0.2369</td>
</tr>
<tr>
<td>Static hedge</td>
<td>0.0761</td>
<td>0.0861</td>
<td>-0.1870</td>
<td>-0.2198</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\pi = 0.7$</td>
<td></td>
</tr>
<tr>
<td>No hedge</td>
<td>0.1362</td>
<td>0.1717</td>
<td>-0.2718</td>
<td>-0.3148</td>
</tr>
<tr>
<td>Static hedge</td>
<td>0.1316</td>
<td>0.1737</td>
<td>-0.2591</td>
<td>-0.2968</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\pi = 1$</td>
<td></td>
</tr>
<tr>
<td>No hedge</td>
<td>0.2520</td>
<td>0.4599</td>
<td>-0.3628</td>
<td>-0.4047</td>
</tr>
<tr>
<td>Static hedge</td>
<td>0.2476</td>
<td>0.4615</td>
<td>-0.3491</td>
<td>-0.3866</td>
</tr>
</tbody>
</table>

Table 7 reports the results for the discounted P&L statistics in presence of static hedge via S-forwards using different levels of equity exposure. One observes that the tail risk measures VaR and the ES are reduced (in absolute terms) when hedging strategy is in place, compared to the situation when there is no hedge. The improvement in percentage terms of the VaR and the ES is larger when the equity exposure is small. The average return decreases slightly when hedging strategy is in place, which suggests that there is a tradeoff between the reduction of a tail risk and the return. The results indicate that a simple static hedge via S-forward is beneficial, especially when the equity exposure of the VA portfolio is small.

8 Conclusion

This paper presents a comprehensive analysis for guaranteed lifetime withdrawal benefits (GLWB) embedded in variable annuities, which have become increasingly popular in recent years. An analysis of GLWB, including product features, identification and quantification of risks is provided. GLWB offer longevity insurance in the form of an income stream covering systematic mortality risk and protecting policyholders from the downside risk of fund investment. The guarantee promises to the insured a stream of income for life, even if the investment account is depleted. After the death of the insured, any savings remaining in the account is returned to the insured’s beneficiary.

The paper shows that GLWB can be priced by two equivalent approaches, based on tractable equity and stochastic mortality models. We study the effect of important financial and demographic variables, including interest rate $r$, volatility of fund investment $\sigma$, volatility of mortality $\sigma_\mu$ and the market price of systematic mortality
risk $\lambda$, on the fair guaranteed fee rate $\alpha^*_g$ charged by the insurer, as well as on the profit and loss characteristics from the point of view of the insurance provider.

The results show that the fair guaranteed fee rate increases with increasing volatility of mortality $\sigma_{\mu}$ and the market price of systematic mortality risk $\lambda$. The fair guaranteed fee rate is also positively related, and highly sensitive to, the equity exposure (hence volatility) of the investment account. The relationship between $\alpha^*_g$ and interest rate $r$ is negative and $\alpha^*_g$ is highly sensitive to low interest rates.

Quantification and assessment of risks underlying GLWB is studied via P&L distributions, assuming no hedging in place. Tail risk measures such as the Value-at-Risk (VaR) and the Expected Shortfall (ES) are higher when systematic mortality risk is present in the model, compared to the situation with no systematic mortality risk. We quantify parameter risk and model risk, showing how these risks could result in significant under- or over-estimation of the fair guaranteed fee rate $\alpha^*_g$. Finally, a static hedging strategy implemented using S-forwards results in the reduction of the VaR and the ES numbers, but also leads to a decrease in the average return for the guarantee provider, especially for low levels of equity exposure. We demonstrate in the paper that while financial risk is substantial for GLWB, the impact of systematic mortality risk on the guarantee cannot be ignored.

Acknowledgement: Fung acknowledges the Australian Postgraduate Award scholarship and financial support from the Australian School of Business, UNSW. Ignatieva acknowledges financial support from the Australian School of Business, UNSW. Sherris acknowledges the support of the Australian Research Council Centre of Excellence in Population and Ageing Research (project number CE110001029).

References


