A Note on the Delta-Hedging Strategy for Variable Annuities

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Abstract

For variable annuities, the cost of hedging must be taken into consideration when firms use the dynamic hedging strategy. In this paper, we study hedging strategies by assuming the hedge position follows a random walk with boundary conditions. We find that re-balancing delta to the initial position is always more cost-efficient than re-balancing it to the edge for a fixed transaction cost. However, when the transaction cost is proportional to the hedge limit, re-balancing to the initial position is always less cost-efficient than re-balancing to the edge. Moreover, we quantify the magnitude of the efficiency in both cases. The results in this paper are may allow practicing actuaries and finance professionals to make judicious decisions.

JEL Classification: G22, G31;

Keywords: Cost of Hedging; Variable Annuities; Re-balancing; Random Walk; Fixed Transaction Cost; Proportional Transaction Cost

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1 Introduction and Literature Review

Variable annuities (VA) are used to describe a broad class of insurance contracts with embedded guarantees. The embedded guarantees can take various forms depending on the benefits of the contracts. For example, the Guaranteed Minimum Maturity Benefit (GMMB) guarantees the policyholders a pre-specified amount of payment on the maturity date of the contract. Other popular types of benefits include Guaranteed Minimum Death Benefit (GMDB), Guaranteed Minimum Income Benefit (GMIB), Guaranteed Minimum Accumulation Benefit (GMAB) and Guaranteed Minimum Withdrawal Benefit (GMWB). For a full description of these contracts, the reader may consult Hardy (2003) or Abbey and Henshall (2007).

The embedded put options poses great challenge for pricing. Equipped with the tools from financial engineering, especially the Black-Scholes formula (See, for example, Hull (2009), Black and Scholes (1973), Merton and Scholes (1973), or Shreve (2005)), insurers can design complex VA products and price them more accurately. However, the risk embedded in the VA contract is still a major concern for all insurance firms. Often times insurance firms use dynamic hedging to reduce the risk. As pointed out by Coleman et al (2006), the literature on VA products tend to focus on pricing rather than hedging. Nonetheless, many important works have been done. Taking the assumptions of the well-known Black-Scholes formula, Hardy (2003) studies the dynamic hedging for several most commonly-used VA products. Under the same Black-Scholes framework, Gupta (2007) gives an upper bound for the costs of the guaranteed minimum withdrawal benefit (GMWB) contracts. Coleman et al (2006) study the problem of hedging VA products under incomplete market with stochastic interest rate. Dai et al (2008) propose a singular stochastic control model for pricing GMWB using. Abbey and Henshall (2007) investigate the cost of delta hedging with hedge limits by assuming the hedge position of the portfolio follows a random walk starting at 0. They argue that it would always be cost-efficient to re-balance delta to the edge of the limit rather than to the initial position 0. Their argument is descriptive and lacks rigorous reasoning. In this paper, we follow Abbey and Henshall (2007) to assume the delta follows a random walk. We study this problem rigorously under two different cost structures:
(I) a fixed transaction cost and (II) a proportional transaction cost. By comparing the long-run costs per unit time associated with the two strategies, we show that, in case I, re-balancing to the edge is always less cost-efficient than re-balancing to the initial position. However, in case II, re-balancing to the edge is always more cost-efficient than re-balancing to the initial position. Thus, the choice between the two strategies depends on the cost structure. We want to emphasize that the results in this paper are applicable to other financial products too. Thus, we believe this paper will be helpful for practicing actuaries and finance experts to make right decisions.

The remaining of this paper is organized as follows: Section 2 gives the notation and setup. In Section 3 we assume a fixed cost per transaction and compare the long-run costs per unit time associated with the two strategies. In Section 4 we investigates the case when the transaction cost is proportional to the hedge limit. Section 5 summarizes our study and concludes the paper.

2 Notation and Setup

In our study, we will assume that the hedge position, delta (∆) of the portfolio, follows a random walk starting at 0, namely, the initial position is 0. We assume the unit of time is one day. We also assume the hedge position is under daily monitor. There are two hedge limits: \( k - 1 \) and \( -(k - 1) \), where \( k \) is a fixed positive integer in the set \{2, 3,...\}. Let \( C \) be the transaction cost, i.e., the cost of re-balancing. We assume there are no other costs. Also, we assume that there are no hedging errors. In our study, we will compare the costs of the two re-balancing strategies in two different cases:

- **Case I**: The transaction cost \( C \) is a fixed amount independent of \( k \),
- **Case II**: The transaction cost \( C \) is proportional to \( k \).

These two cases corresponds to the two common cost structures in securities trading: (I) a fixed cost per transaction and (II) the cost proportional to the transaction amount. For simplicity, we call the strategy of re-balancing \( ∆ \) to 0 **strategy 1** and the strategy of re-balancing \( ∆ \) to the edge **strategy 2**. For \( i = 1, 2 \), we will use \( C_i \) to denote the long-run cost
per unit time for strategy $i$. We will use $T_1$ to denote the number of days for $\Delta$ to reach either $k$ or $-k$ starting at 0 for strategy 1. Similarly, We will use $T_2$ to denote the number of days for $\Delta$ to reach either $k$ or $-k$ starting at $k-1$ or $-(k-1)$ for strategy 2. The goal of our study is to compare the two strategies in terms of long-run cost per unit time.

In both cases I and II, $\Delta$ will be re-balanced immediately once it exceeds the hedge limits. Note that no action will be taken if $\Delta$ is on the edge $k-1$ or $-(k-1)$. In other words, $\Delta$ will be re-balanced once it hits either $k$ or $-k$.

### 3 Fixed Transaction Cost

We will use $p$ to denote the up-state probability, i.e., the probability that $\Delta$ moves upwards during a day. Similarly, we let $q = 1 - p$ denote the down-state probability. Define

$$p_{1,k} \equiv P\{\Delta \text{ starts at 0 and reaches } k \text{ before it hits } -k\},$$

and

$$q_{1,k} \equiv 1 - p_{1,k} = P\{\Delta \text{ starts at 0 and reaches } -k \text{ before it hits } k\}.$$

**Remark.** The first subscript 1 denotes strategy 1 and the second subscript $k$ denotes the starting point in the gambler’s ruin problem, which will be explained shortly.

To calculate $p_{1,k}$, we can turn resort to the well-know gambler’s ruin problem. (See, for example, Ross (1995).) In terms of gambler’s ruin problem, $p_{1,k}$ equals the probability that the gambler starts with a fortune of $k$ and his fortune will reaches $2k$ before reaching 0. By the well-known results from the gambler’s ruin problem, we have

$$p_{1,k} = \frac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^{2k}},$$

$$q_{1,k} = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^{2k}}{1 - \left(\frac{q}{p}\right)^{2k}},$$

(1)
and
\[ E[T_1] = \frac{1}{2p-1} \left\{ \frac{2k}{1 - \left( \frac{q}{p} \right)^{2k}} - k \right\}. \tag{2} \]

For strategy 1, we say a cycle is completed if \( \Delta \) starts at 0 and is re-balanced to 0 for the first time. Note that within each cycle, transaction cost occurs exactly once.

For strategy 2, things are a little bit different. If \( \Delta \) hits \( k \), then it will be re-balanced to \( k - 1 \). On the other hand, if it hits \( -k \), it will be re-balanced to \( -(k - 1) \). Define

\[
\begin{align*}
p_{2,k-1} &\equiv P\{\Delta \text{ starts at } k - 1 \text{ and reaches } k \text{ before it hits } -k\}, \\
q_{2,k-1} &\equiv 1 - p_{2,k-1} = P\{\Delta \text{ starts at } k - 1 \text{ and reaches } -k \text{ before it hits } k\}, \\
p_{2,1} &\equiv P\{\Delta \text{ starts at } -(k - 1) \text{ and reaches } k \text{ before it hits } -k\}, \\
q_{2,1} &\equiv 1 - p_{2,1} = P\{\Delta \text{ starts at } -(k - 1) \text{ and reaches } -k \text{ before it hits } k\}.
\end{align*}
\]

Then \( p_{2,k-1}, q_{2,k-1}, p_{2,1} \) and \( q_{2,1} \) can also be interpreted in terms of gambler’s ruin problem. Thus, we have

\[
\begin{align*}
p_{2,k-1} &= 1 - \left( \frac{q}{p} \right)^{2k-1}, \\
q_{2,k-1} &= \frac{\left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k}}{1 - \left( \frac{q}{p} \right)^{2k}}, \\
p_{2,1} &= \frac{1 - \left( \frac{q}{p} \right)^{2k}}{1 - \left( \frac{q}{p} \right)^{2k}}, \\
q_{2,1} &= \frac{\left( \frac{q}{p} \right) - \left( \frac{q}{p} \right)^{2k}}{1 - \left( \frac{q}{p} \right)^{2k}}.
\end{align*}
\tag{3}
\]

Equipped with the above notation, we can describe the stochastic behavior of \( \Delta \) for strategy 2. For strategy 2, \( \Delta \) starts at \( t = 0 \) as in strategy 1. It hits \( k \) first with probability \( p_{1,k} \), or hits \( -k \) first with probability \( q_{1,k} \). After \( \Delta \) hits \( k \) for the first time,
it is re-balanced to $k - 1$. After that, it will hit $k$ first with probability $p_{2,2k-1}$, or hit $-k$ first with probability $q_{2,2k-1}$. If $\Delta$ starts at 0 and hits $-k$ for the first time, it is re-balanced to $-(k - 1)$. After that, it will hit $k$ first with probability $p_{2,1}$, or hit $-k$ first with probability $q_{2,1}$. The stochastic behavior of $\Delta$ after it hits $k$ or $-k$ for the very first time can be modeled as a regenerative process. Thus, for strategy 2, we say a cycle is completed if $\Delta$ starts at either $k - 1$ or $-(k - 1)$ and is re-balanced to $k - 1$ or $-(k - 1)$ for the first time. Similar to strategy 1, transaction cost occurs exactly once within each cycle.

Since $E[T_i]$ is the expected length of cycle for strategy $i, (i = 1, 2)$, Wald’s first identify implies that the expected number of cycles per unit time for strategy $i$ equals $\frac{1}{E[T_i]}$. Thus, for a fixed cost $C$ per transaction, the ratio of $C_1$ to $C_2$ equals

$$\frac{C_1}{C_2} = \frac{C}{E[T_1]} \frac{1}{E[T_2]} = \frac{E[T_2]}{E[T_1]}.$$  \hspace{1cm} (4)

Note that the stochastic behavior of $\Delta$ from the moment it starts at 0 until it hits $k$ or $-k$ for the first time is the same for both strategy 1 and strategy 2. Thus, we may compare the two strategies from the moment it hits $k$ or $-k$ for the very first time. Also, the symmetric random walk is recurrent, the ratio in equation (4) will be meaningless. Therefore, with loss of generality, we will always assume that $\Delta$ follows a non-symmetric random walk.

Now we make a crucial observation for strategy 2: if we focus only on the epoches $\Delta$ is re-balanced to $k - 1$ or $-(k - 1)$, then we will identity an embedded Markov chain with a state space $S = \{- (k - 1), k - 1\}$. Note that we do not track the moments $\Delta$ hits $k - 1$ starting from $-(k - 1)$, or $\Delta$ hits $-(k - 1)$ starting from $k - 1$. Each transition of this Markov chain actually corresponds to a cycle defined for strategy 2. Without loss of generality, we may call state $k - 1$ state 1 and state $-(k - 1)$ state 2. Then the transition matrix of this Markov chain is given by

$$P = \begin{pmatrix} p_{2,2k-1} & q_{2,2k-1} \\ p_{2,1} & q_{2,1} \end{pmatrix}.$$  \hspace{1cm} (5)
Clearly, this Markov chain is irreducible, aperiodic and positive recurrent. Thus, we can find the unique solution to the system of ergodic equations

\[
\pi_j = \sum_{i=1}^{2} \pi_i P_{ij}, \quad (j = 1, 2),
\]

\[
\sum_{j=1}^{2} \pi_j = 1.
\]

It’s easy to see that the solution to this system is given by

\[
\pi_1 = \frac{P_{21}}{1 - P_{11} + P_{21}},
\]

\[
\pi_2 = \frac{1 - P_{11}}{1 - P_{11} + P_{21}}.
\]

By equations (3), (5) and (6), we find that

\[
\pi_1 = \frac{1 - \left( \frac{q}{p} \right)}{1 - \left( \frac{q}{p} \right) + \left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k}},
\]

\[
\pi_2 = \frac{\left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k}}{1 - \left( \frac{q}{p} \right) + \left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k}}.
\]

To find \( E[T_2] \), we condition on the state \( \Delta \) will be in over the long run:

\[
E[T_2] = \pi_1 E[T_{k-1}] + \pi_2 E[T_{-(k-1)}],
\]

where \( T_{k-1} \) denotes the number of days for \( \Delta \) to reach \( k \) or \(-k\) starting at \( k - 1 \), and \( T_{-(k-1)} \) denotes the number of days for \( \Delta \) to reach \( k \) or \(-k\) starting at \(-(k - 1)\). Using the results from gambler’s ruin problem, we have

\[
E[T_{k-1}] = \frac{1}{2p - 1} \left\{ \frac{2k \left[ 1 - \left( \frac{q}{p} \right)^{2k-1} \right]}{1 - \left( \frac{q}{p} \right)^{2k}} - (2k - 1) \right\},
\]

\[
E[T_{-(k-1)}] = \frac{1}{2p - 1} \left\{ \frac{2k \left[ 1 - \left( \frac{q}{p} \right) \right]}{1 - \left( \frac{q}{p} \right)^{2k}} - 1 \right\}.
\]

Indeed, \( E[T_{k-1}] \) and \( E[T_{-(k-1)}] \) are expected sojourn times in states 1 and 2 respectively.
Putting equations (1) and (8) into equation (7), we obtain

\[
E[T_2] = \left( \frac{1}{2p-1} \right) \left[ \frac{1 - \left( \frac{q}{p} \right)}{1 - \left( \frac{q}{p} \right) + \left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k}} \right] \left\{ 2k \left[ 1 - \left( \frac{q}{p} \right)^{2k-1} \right] - (2k - 1) \right\}
+ \left( \frac{1}{2p-1} \right) \left[ \frac{\left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k}}{1 - \left( \frac{q}{p} \right) + \left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k}} \right] \left\{ 2k \left[ 1 - \left( \frac{q}{p} \right) \right] - 1 \right\}
\]

(9)

Next, we use equations (2) and (9) to get

\[
\frac{E[T_2]}{E[T_1]} = \frac{N}{D}
\]

where

\[
N = \left[ 1 - \left( \frac{q}{p} \right) \right] \left\{ 2k \left[ 1 - \left( \frac{q}{p} \right)^{2k-1} \right] - (2k - 1) \left[ 1 - \left( \frac{q}{p} \right)^{2k} \right] \right\}
+ \left[ \left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k} \right] \left\{ 2k \left[ 1 - \left( \frac{q}{p} \right) \right] - 1 - \left( \frac{q}{p} \right)^{2k} \right\},
\]

and

\[
D = \left[ 1 - \left( \frac{q}{p} \right) + \left( \frac{q}{p} \right)^{2k-1} - \left( \frac{q}{p} \right)^{2k} \right] \left\{ 2k \left[ 1 - \left( \frac{q}{p} \right)^{k} \right] - k \left[ 1 - \left( \frac{q}{p} \right)^{2k} \right] \right\}.
\]

After some algebra, we have

\[
N = \left[ 1 - \left( \frac{q}{p} \right) \right] \left[ 1 - \left( \frac{q}{p} \right)^{2k-1} \right] \left[ 1 - \left( \frac{q}{p} \right)^{2k} \right],
\]

\[
D = k \left[ 1 - \left( \frac{q}{p} \right) \right] \left[ 1 + \left( \frac{q}{p} \right)^{2k-1} \right] \left[ 1 - \left( \frac{q}{p} \right)^{k} \right]^2.
\]

Therefore,

\[
\frac{C_1}{C_2} = \frac{E[T_2]}{E[T_1]} = \frac{\left[ 1 + \left( \frac{q}{p} \right)^k \right] \left[ 1 - \left( \frac{q}{p} \right)^{2k-1} \right]}{k \left[ 1 - \left( \frac{q}{p} \right)^k \right] \left[ 1 + \left( \frac{q}{p} \right)^{2k-1} \right]}.
\]

(10)

It turns out that the right-hand side of equation (10) is bounded from above by 0.75.

We will state this fact as a theorem. To do this, we first need to prove a lemma.
Lemma 3.1 Suppose $k \in \{2, 3, \ldots \}$. Consider the function

$$f(p) = \frac{1 + \left(\frac{2}{p}\right)^k}{k \left[ 1 - \left(\frac{2}{p}\right)^k \right]} \left[ 1 - \left(\frac{2}{p}\right)^{2k-1} \right],$$

where $q = 1 - p$. Then $f$ is increasing on the interval $(0, \frac{1}{2})$ and decreasing on the interval $(\frac{1}{2}, 1)$.

Proof. Note that $f$ can be rewritten as

$$f(p) = \frac{1}{k} \times \frac{1 - \left(\frac{1}{p} - 1\right)^{3k-1}}{1 - \left(\frac{1}{p} - 1\right)^{3k-1}} + \frac{\left(\frac{1}{p} - 1\right)^k - \left(\frac{1}{p} - 1\right)^{2k-1}}{1 - \left(\frac{1}{p} - 1\right)^{3k-1}} - \frac{\left(\frac{1}{p} - 1\right)^k - \left(\frac{1}{p} - 1\right)^{2k-1}}{1 - \left(\frac{1}{p} - 1\right)^{3k-1}}.$$

First, we examine $f$ on the interval $(0, \frac{1}{2})$. To this end, define $h(p) = \left(\frac{1}{p} - 1\right)^k - \left(\frac{1}{p} - 1\right)^{2k-1}$, then we have

$$f(p) = \frac{1}{k} \times \frac{1 - \left(\frac{1}{p} - 1\right)^{3k-1}}{1 - \left(\frac{1}{p} - 1\right)^{3k-1}} + h(p)$$
$$- \frac{h(p)}{1 - \left(\frac{1}{p} - 1\right)^{3k-1}}.$$ 

(11)

Moreover, the derivative of $h$ is given by

$$h'(p) = \frac{1}{p^2} \left[ (2k - 1) \left(\frac{1}{p} - 1\right)^{2k-2} - k \left(\frac{1}{p} - 1\right)^{k-1} \right],$$

which is clearly positive on $p \in (0, \frac{1}{2})$. Thus, $h$ is increasing on $(0, \frac{1}{2})$. It follows from (11) that $f$ is also increasing on $(0, \frac{1}{2})$. Using the same argument, we can easily show that $f$ is decreasing on $(\frac{1}{2}, 1)$. Hence the lemma follows.

Now we can prove the following theorem.

Theorem 3.2 For a fixed transaction cost, re-balancing to 0 is always more cost-efficient than re-balancing to the edge. Moreover, the ratio of the long-run costs per unit associated with the two strategies, given by equation (10), is bounded from above by 0.75 for any positive integer $k$ in the set $\{2, 3, \ldots \}$ and up-state probability $p \neq \frac{1}{2}$. 

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Proof. By Lemma 3.1, we only need to find the limit of \( f \) as \( p \to \frac{1}{2} \) provided the limit exists. Using L'Hôpital's rule, we have

\[
\lim_{p \to \frac{1}{2}} f(p) = \frac{1}{k} \times \left( \frac{\left(1 - \left(\frac{1}{p} - 1\right)^{3k-1}\right) + \left(\left(\frac{1}{p} - 1\right)^k - \left(\frac{1}{p} - 1\right)^{2k-1}\right)}{\left(1 - \left(\frac{1}{p} - 1\right)^{3k-1}\right) - \left(\left(\frac{1}{p} - 1\right)^k - \left(\frac{1}{p} - 1\right)^{2k-1}\right)} \right)'
\]

\[
= \frac{2k - 1}{k^2}
\]

\[
= 1 - \left(1 - \frac{1}{k}\right)^2.
\]

Since \(1 - \left(1 - \frac{1}{k}\right)^2\) is decreasing in \(k\) on the set \(\{2, 3, \ldots\}\), we have

\[
\frac{C_1}{C_2} = \left[1 + \left(\frac{q}{p}\right)^k\right] \left[1 - \left(\frac{q}{p}\right)^{2k-1}\right] \leq 1 - \left(1 - \frac{1}{k}\right)^2. \quad (12)
\]

From (11), it's clear that \(C_1/C_2\) reaches its maximum value 0.75 when \(k = 2\). This completes the proof.

Theorem 3.2 not only shows that in the case of a fixed transaction cost, strategy 1 is always more cost-efficient than strategy 2, it also quantifies the magnitude of the efficiency. In other words, the long-run cost per unit time associated with strategy 1 is at most 75% of that associated with strategy 2. In addition, equation (12) shows that as the edge of the limit increases, the upper bound of \(C_1/C_2\) is getting smaller and smaller. Consequently, strategy 1 will be more and more cost-efficient than strategy 2 as \(k\) increases.

Example 1. Table 1 below gives some values of \(C_1/C_2\) for different values of \(k\) and \(p\) accurate to three digits. From the table, we make the following observations.

- For a fixed \(p\), the value of \(C_1/C_2\) decreases as \(k\) increases from 2 to 10.

- \(C_1/C_2\) increases as \(p\) goes from 0.1 to 0.45, then it decreases as \(p\) goes from 0.55 to 0.9.
• The table shows a symmetric pattern: for a given $k$, the values of $C_1/C_2$ at $p$ and $1-p$ are the same. For example, when $k = 2$, the value of $C_1/C_2$ at $p = 0.45$ equals value of $C_1/C_2$ at $p = 0.55$. Intuitively, this makes sense because $\Delta$ can exceed two edge limits: $k-1$ and $-(k-1)$, which are symmetric with respect to $x$-axis. If we reflect everything with respect to $x$-axis, and rename the edge if necessary, then the associated costs should be the same when the up-probability equals $p$ and $1-p$.

• All these numerical values are consistent with Lemma 3.1 and Theorem 3.2.

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<td>0.125</td>
<td>0.103</td>
<td>0.100</td>
<td>0.100</td>
<td>0.100</td>
</tr>
</tbody>
</table>

4 Proportional Transaction Cost

In this case, we assume the transaction cost is proportional to the transaction amount. Thus, the transaction cost per cycle is $k$ for strategy 1 and 1 for strategy 2. It’s clear that the stochastic behavior of $\Delta$ is the same as that in Case I. Hence by equation (10), we know that the ratio of the long-run transaction costs associated with the two strategies
is given by

\[
\frac{C_1}{C_2} = \frac{1 \times \frac{1}{E[T_1]}}{\frac{1}{E[T_2]}} = k \frac{E[T_2]}{E[T_1]} = \left[ \frac{1 + \left( \frac{2}{p} \right)^k}{1 - \left( \frac{2}{p} \right)^k} \right] \cdot \left[ \frac{1 - \left( \frac{2}{p} \right)^{2k-1}}{1 + \left( \frac{2}{p} \right)^{2k-1}} \right].
\]

(13)

Note that equations (10) and (13) differ only by a multiple \( \frac{1}{k} \). The next theorem compares the cost-efficiencies of the two strategies for a proportional transaction cost.

**Theorem 4.1** For a proportional transaction cost, re-balancing to 0 is always less cost-efficient than re-balancing to the edge. Moreover, for a fixed positive integer \( k \) in the set \( \{2, 3, \ldots\} \), the ratio of the long-run costs per unit associated with the two strategies, given by equation (13), is bounded from above by \( \leq 2 - \frac{1}{k} \) for any up-state probability \( p \neq \frac{1}{2} \).

**Proof.** We can use the same argument used in the proof of Lemma 3.1 to show that the right-hand side of equation (13) is increasing on the interval \((0, \frac{1}{2})\) and decreasing on the interval \((\frac{1}{2}, 1)\). Then we follow the same line of reasoning to conclude that

\[
\frac{C_1}{C_2} = \left[ \frac{1 + \left( \frac{2}{p} \right)^k}{1 - \left( \frac{2}{p} \right)^k} \right] \cdot \left[ \frac{1 - \left( \frac{2}{p} \right)^{2k-1}}{1 + \left( \frac{2}{p} \right)^{2k-1}} \right] \leq 2 - \frac{1}{k}.
\]

(14)

It follows that \( C_1/C_2 \) reaches its maximum value 1.5 when \( k = 2 \). Thus, the theorem is established.

Theorem 4.1 shows that, in the case of a proportional transaction cost, strategy 2 is always more cost-efficient than strategy 1. As pointed out by Abbey and Henshall (2007), in this case \( \Delta \) may move closer to 0 without the extra transaction costs, i.e., the transaction cost of re-balancing delta from \( k - 1 \) or \(-(k - 1)\) to 0. (Note that this extra transaction cost vanishes in the case of a fixed transaction cost.) The magnitude of the efficiency is also quantified by equation (14). Although the the upper-bound \( \leq 2 - \frac{1}{k} \) increases in \( k \), equation (14) cannot ensure that strategy 2 will be more and more cost-efficient than strategy 1 as the hedge limit increases. This is due to the fact that
$C_1/C_2$ can decreases when its upper bound increases. Note this observation is different from its counterpart in Case I.

**Example 2.** Table 2 below provides some numerical values of $C_1/C_2$ for different values of $k$ and $p$. From the table, we can see the following facts.

- For most values of $p$, the value of $C_1/C_2$ decreases when $k$ increases from 2 to 10. However, when $p = 0.45$ and $p = 0.55$, the values of $C_1/C_2$ first increases and then decreases as $k$ goes from 2 to 10. Thus, it’s possible that the value of $C_1/C_2$ decreases when $k$ increases, even if the upper bound of $C_1/C_2$ increases.

- $C_1/C_2$ first increases as $p$ goes from 0.1 to 0.45, then it decreases as $p$ goes from 0.55 to 0.9.

- There is a symmetric pattern as in Case I: for a given $k$, the values of $C_1/C_2$ at $p$ and $1-p$ are the same.

- All the numerical values are consistent with Theorem 4.1 and our discussion above.

![Table 2: Values of $C_1/C_2$ for Different Values of $k$ and $p$.](image)}
5 Summary

In this paper, we study the long-run costs per unit time associated with two hedging strategies for variable annuities when $\Delta$ is assumed to follow a random walk with boundary conditions. The first strategy is to re-balance $\Delta$ to the initial position 0 when it exceeds the hedging limit, while the second strategy is to re-balance it to the edge of the limit. Under two popular trading cost structures, we derive the explicit formulas for the long-run costs per unit time associated with the two strategies. We find that re-balancing to 0 is always more cost-efficient than re-balancing to the edge for a fixed transaction cost. However, in the case of a proportional transaction cost, we show that re-balancing to the edge of the limit will always be more cost-efficient than re-balancing to 0. The results of this paper may be helpful for both practicing actuaries and finance experts to make judicious choices between the two strategies.

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References


