Improving skewness of mean-variance portfolios

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Abstract

The widely accepted belief that asset returns and insurance product line margins are not normally distributed has motivated the use of skewness (or higher than second order moments), in the context of optimal risk-reward portfolio allocation. Here, we propose an optimization-based methodology to substantially improve the skewness of portfolios in the mean-variance efficient frontier. Unlike other related methods, the proposed methodology is very intuitive, non-iterative, simple to implement, and it can be readily and efficiently carried out using state of the art optimization solvers. These characteristics should be very appealing to risk managers.

Keywords: mathematical methods and programming (C6); risk management; portfolio allocation; VaR constraint; asymmetric returns.

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1 Introduction

In recent years, there has been a growing interest in developing methodologies to take into account the exposure to high losses, and moments of higher order than the variance, in the context of optimal risk-reward portfolio allocation. This is due to the widely accepted belief that asset returns and insurance product line margins are not normally distributed, and their inter-relationships cannot be characterized only by their correlations. Mainly, these methodologies aim at controlling the characteristics of the tails of the portfolio return distribution; specially the left tail (losses).

Let us avoid going through a lengthy description of the extensive related literature by concentrating on two particular methodologies; namely, the ones developed by Boyle and Ding (2005); and Konno et al. (1993). In both these methodologies, the aim is to improve the skewness of portfolios in the mean-variance efficient frontier (cf. Markowitz (1952)); while maintaining the resulting portfolio close to the mean-variance efficient frontier. These methodologies also have the characteristic that no distributional assumption about the assets return distribution is required. In particular, the methodologies’ parameters can be obtained from historical samples of the assets returns. Clearly, the parameters can also be obtained by sampling the assets returns from a desired model or distribution.

Along the lines of the work of Boyle and Ding (2005); and Konno et al. (1993), we propose here an optimization-based methodology to substantially improve the skewness of portfolios in the mean-variance efficient frontier; while maintaining the resulting portfolio close to the mean-variance efficient frontier. Unlike other related methods in the literature, the proposed methodology is:

- **very intuitive**: unlike the Boyle and Ding (2005); and Konno et al. (1993) methodologies that (loosely speaking) require setting a number of ad-hoc parameters related to how the skewness is approximated, the methodology presented here requires a single extra parameter in comparison to the classical mean-variance model that, as it will be shown, can be easily and intuitively set-up.

- **non-iterative**, unlike the Boyle and Ding (2005); and Konno et al. (1993) method-
ologies that require a number of iterations of the methodology to obtain the desired portfolio, the methodology presented here requires a single iteration.

- *simple to implement and carry out:* the methodology presented here is as simple to implement and carry out as a mean-variance model with (linear) diversification constraints. Thus the methodology can be carried out using readily available optimization or financial commercial software such as Excel Solver, ILOG-CPLEX, Matlab, BARRA, to name just a few.

These characteristics should make our methodology very appealing to both practitioners and researchers in risk and portfolio management; specially for those interested in using simple tools to control the tail distributional characteristics of their portfolios, within the classical mean-variance framework. For example; portfolio allocation models such as the classical mean-variance portfolio model (and its variations), or as the Conditional Value-at-Risk (CVaR) of Rockafellar and Uryasev (2000) (and its variations); which have these characteristics, are amongst the most popular portfolio allocation models within practitioners and researchers.

As stated before, the methodology presented here does not require any distributional assumptions, and aims at controlling the left tail (high losses) of the portfolio return distribution. Thus, the methodology should be of special interest to actuaries, who face portfolio allocation problems over assets for which very sparse historical data is available (e.g., sometimes only quarterly data is available), and that due to regulations and the importance of catastrophic events in actuarial science, are interested in controlling the left tail (high losses) of the portfolio return distribution. Although for ease of data gathering, our examples in the paper focus on portfolios of assets, we note that our methodology applies directly to portfolios of assets and liabilities (considered as assets with negative returns) that are typical in actuarial science (see, e.g., Griffin and Boomgaardt (1999)). In fact, the methodology presented here is the result of a collaboration with a leading US Asset/Liability management consultant company.

The optimization model at the basis of the methodology is obtained by adding a
simple probability constraint to the classical mean-variance portfolio allocation model (cf. Markowitz (1952)). The addition of probability constraints to the classical mean-variance portfolio allocation model is by no means a novel idea, and has been extensively studied in the literature. Our contribution is to show that such type of constraint can be used in a “means to an end” fashion, to in a simple way improve the skewness of portfolios in the mean-variance efficient frontier.

The article is organized as follows. In Section 2, we formally present the proposed optimization model that will be used to improve the skewness of portfolios in the mean-variance efficient frontier. In Section 3, we describe the main characteristics of the experiments that will be used to illustrate the proposed methodology and its characteristics. Those experiments are presented in Section 4, where we show how the skewness of portfolios in the mean-variance efficient frontier can be improved; while maintaining the resulting portfolio close to the efficient frontier. We conclude in Section 5, with some final remarks and directions for future work.

2 The optimization model

In this section, we formally state the optimization model that will be used to substantially improve the skewness of portfolios in the mean-variance efficient frontier (cf. Markowitz (1952)); while maintaining the resulting portfolio close to the mean-variance efficient frontier. For that purpose, we begin by presenting the classical mean-variance portfolio (allocation) model (cf. Markowitz (1952)).

Consider $n$ risky assets that can be chosen by an investor in the financial market. Let $r = (r_1, \ldots, r_n)^T \in \mathbb{R}^n$ denote the uncertain returns of the $n$ risky assets from the current time $t = 0$ to a fixed future time $t = T$. Let $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ denote a portfolio on these assets; that is, the percentage of the available funds to be allocated in each of the $n$ risky assets. Thus, the portfolio return from $t = 0$ to $t = T$ is given by

$$r^Tx = x^Tr = \sum_{i=1}^{n} x_ir_i.$$
A (single-period) mean-variance portfolio model aims at finding the portfolio $x$ to be constructed at $t = 0$, in order to minimize the variance of the portfolio returns, subject to the portfolio having a given minimum expected return $\mu_0$, and possibly satisfying some linear diversification constraints. Formally, the mean-variance portfolio model can be written as the following optimization problem:

$$\begin{align*}
\min & \quad \text{Var}(x^T r) \\
\text{s.t.} & \quad \mathbb{E}(x^T r) \geq \mu_0 \\
& \quad x^T \mathbf{1} = 1 \\
& \quad x \in \mathcal{X} \subseteq \mathbb{R}_+^n,
\end{align*}$$

where $\mathbf{1} \in \mathbb{R}^n$ is the vector of all-ones, $x^T \mathbf{1} = \sum_{i=1}^n x_i$, $\mu_0 \in \mathbb{R}$ is the given minimum expected portfolio return, and $\mathcal{X} \subseteq \mathbb{R}_+^n$ is a given set defined by linear constraints; which are typically used to enforce certain diversification constraints on the portfolio $x$. (Here, $\text{Var}(\cdot)$, $\mathbb{E}(\cdot)$, and $\mathbb{P}(\cdot)$, respectively denote variance, expectation, and probability.) For the moment, we will assume that no short positions are allowed in the portfolio (i.e., $\mathcal{X} \subseteq \mathbb{R}_+^n$); which is the most common situation in practice (cf. Michaud (1998)). In Section 5, we will consider the case in which short positions are allowed in the portfolio.

The main approach to solve (1) is the so-called sampling approach, which uses a finite number of samples $r_1^T, \ldots, r_m^T \in \mathbb{R}^n$ of the asset returns, that are typically obtained from historical data, simulations, or a combination of both. Here, we adopt the sampling approach, which leads to the mean-variance portfolio model (1) being written as:

$$\begin{align*}
z_{\text{MV}} := \min & \quad x^T \sum x = \sum_{i=1}^n \sum_{j=1}^n x_i \sigma_{ij} x_j \\
(\text{MV}) & \quad \text{s.t.} \quad x^T \mu \geq \mu_0 \\
& \quad x^T \mathbf{1} = 1 \\
& \quad x \in \mathcal{X} \subseteq \mathbb{R}_+^n,
\end{align*}$$

where the vector $\mu \in \mathbb{R}^n$ of mean asset returns estimates is typically obtained by
letting

\[ \mu = \frac{1}{m} \sum_{j=1}^{m} r^j, \]  

(3)

and \( \sigma_{ij} \) is the \( ij \)-th element of the matrix \( \Sigma \) of variance-covariance estimates of the asset returns is typically obtained by letting

\[ \Sigma = \frac{1}{m-1} \sum_{j=1}^{m} (r^j - \mu)^T (r^j - \mu). \]  

(4)

Although (for simplicity) in our numerical experiments we will use (3) and (4) to obtain \( \mu \) and \( \Sigma \); our approach is independent of this choice, and a variety of other estimation methods can be used. In particular, estimation methods recently proposed to improve the robustness of mean-variance portfolios, or to make them more intuitively correct in terms of investor’s views, can be readily used within the framework of the proposed methodology. More details about the flexibility of our approach will be given in Section 5.

Here we propose an optimization-based methodology to substantially improve the skewness of portfolios in the mean-variance efficient frontier; that is, portfolios that are obtained by solving the mean-variance portfolio allocation model (1). Formally, the proposed methodology finds portfolios that, in comparison with portfolios in the mean-variance efficient frontier, have significantly better skewness, and at the same time are very close to the mean-variance efficient frontier. These characteristics will be illustrated in Section 4.

The optimization model at the basis of the methodology is obtained by adding a simple probability constraint to (1). Specifically, given \( \alpha \in \mathbb{R} \), we add a constraint limiting the probability \( \mathbb{P}(x^T r \geq \alpha) \) of the portfolio return being greater than or equal to \( \alpha \) to be equal to 1. After adding the probability constraint; the mean-variance
portfolio model (1) becomes:

\[
\begin{align*}
\text{min} & \quad \text{Var}(x^T r) \\
\text{s.t.} & \quad \Pr(x^T r \geq \alpha) = 1 \\
& \quad \mathbb{E}(x^T r) \geq \mu_0 \\
& \quad x^T \mathbf{1} = 1 \\
& \quad x \in X \subseteq \mathbb{R}^n_+.
\end{align*}
\]

(5)

Notice that if we (for example) let \( \alpha = \mu_0 - \kappa \) for an appropriate \( \kappa \in \mathbb{R}_+ \), then the first constraint in (5) forces the portfolio return distribution to have a probability of incurring on a “loss” higher (or a portfolio return lower) than \( \mu_0 - \kappa \) to be zero. Thus, the constraint \( \Pr(x^T r \geq \alpha) = 1 \) in (5) can be used to control the exposure of the portfolio to high losses or high underperformance. As mentioned before, here we will use this constraint in a “means to an end” fashion, to improve the skewness of the portfolio returns distribution.

As with the mean-variance model, we consider the sampling approach to address (5). Namely, we rewrite (5) as:

\[
\begin{align*}
z_{MV^*} := & \quad \text{min} & \quad x^T \Sigma x \\
\text{s.t.} & \quad x^T r^j \geq \alpha & j = 1, \ldots, m \\
& \quad x^T \mu \geq \mu_0 \\
& \quad x^T \mathbf{1} = 1 \\
& \quad x \in X \subseteq \mathbb{R}^n_+,
\end{align*}
\]

(MV*)

(6)

where \( \mu, \mu_0, \Sigma, \) and \( X \) are the same ones used in (2). Notice that the first set of constraints ensures that the probability constraint is satisfied, as all the sample portfolio returns are required to be greater than or equal to \( \alpha \). Moreover, notice that (6) is a convex Quadratic Program of the same class as (2) (i.e., both problems have convex quadratic objectives with linear constraints). Thus (6) can be solved using readily available optimization or financial commercial software such as Excel Solver, ILOG-CPLEX, Matlab, BARRA, to name just a few.

From now on, we will refer to the classical mean-variance portfolio model (2) as the MV model, and the modified MV model (6), as the MV* model. Correspondingly,
we will refer to a portfolio that solves (2) as a MV portfolio, and to a portfolio that solves (6) as a MV* portfolio.

3 About the Numerical Experiments

In Section 4, we will present the results of relevant numerical experiments that focus on illustrating the MV* methodology and its characteristics. Namely, we will focus on showing how the MV* model can be used to find portfolios that, in comparison with MV portfolios, have significantly better VaR and skewness, and at the same time are very close to the mean-variance efficient frontier. Also, given that the MV* model is in general inherently difficult to solve, we will also focus on showing that state of the art optimization solvers can quickly solve practically relevant instances of the MV* model.

Before stating the results of these experiments, we describe some of their common characteristics. Unless otherwise specified, in all the experiments we choose $\mathcal{X} = \mathbb{IR}^n_+$ (i.e., no short selling is allowed) We use ILOG–CPLEX with its default settings to solve all the corresponding Quadratic Programs resulting from the MV and MV* models.

When solving (1), a common practice (for both researchers and practitioners) is to use 60 to 120 months of historical observations of the asset returns to construct the portfolio allocation model (see, e.g., Ceria and Stubbs (2006); DeMiguel et al. (2007); Konno and Yamazaki (1991); Konno and Wijayanayake (2001); Black and Litterman (2001); Boyle and Ding (2005); and Bachman (2007)). Therefore, unless otherwise specified, we consider instances of the MV model and its corresponding MV* model, in which the number of samples $m = 120$. In particular, we will consider finding the optimal MV and MV* portfolio composed of eleven (11) industry portfolios. For that purpose we sample monthly returns of the following 11 industry portfolios:
Historical monthly returns between January 1997 and December 2006 are taken from Ken French’s website http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/; specifically from data file 38\_Industry\_Portfolios.zip. As mentioned earlier, the vector of mean estimates $\mu$, and the matrix of variance-covariance estimates $\Sigma$, of the asset returns are obtained using (3) and (4). This particular choice of data set and estimators ensures that our results can be easily reconstructed by other researchers. Note that Ken French’s website is one of the most recognized sources of financial data for academic purposes. In Section 5, we discuss other possible choices of estimators for $\mu$, and $\Sigma$. The author’s will be happy to provide the reader with both the data sets and programs used to obtain the results presented in the next section. Summary statistical information (mean, variance-covariance, skewness) of the 11 industry portfolios is presented in Table 1.

### 4 Improving skewness

In this section, we concentrate on illustrating how the MV$^*$ model can be used to improve the skewness of MV portfolios. For that purpose, we consider the MV and
Table 1: Summary statistical data (mean, variance-covariance, skewness) of the 11 industry portfolios. Estimates based on monthly returns between January 1997 and December 2006.

<table>
<thead>
<tr>
<th>Mean Return</th>
<th>Agric</th>
<th>Stone</th>
<th>Food</th>
<th>Smoke</th>
<th>Paper</th>
<th>Print</th>
<th>Chems</th>
<th>Prlm</th>
<th>Lethr</th>
<th>Utils</th>
<th>Retail</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0123</td>
<td>1.4990</td>
<td>0.6524</td>
<td>1.5187</td>
<td>0.8668</td>
<td>0.9106</td>
<td>0.7665</td>
<td>1.2995</td>
<td>1.7558</td>
<td>0.9943</td>
<td>1.0532</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variance-Covariance</th>
<th>Agric</th>
<th>Stone</th>
<th>Food</th>
<th>Smoke</th>
<th>Paper</th>
<th>Print</th>
<th>Chems</th>
<th>Prlm</th>
<th>Lethr</th>
<th>Utils</th>
<th>Retail</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Skewness</th>
<th>Agric</th>
<th>Stone</th>
<th>Food</th>
<th>Smoke</th>
<th>Paper</th>
<th>Print</th>
<th>Chems</th>
<th>Prlm</th>
<th>Lethr</th>
<th>Utils</th>
<th>Retail</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7442</td>
<td>0.1295</td>
<td>-0.2099</td>
<td>-0.1571</td>
<td>0.5869</td>
<td>0.1974</td>
<td>-0.3887</td>
<td>0.5397</td>
<td>-0.5583</td>
<td>-0.3907</td>
<td>-0.1978</td>
<td></td>
</tr>
</tbody>
</table>

MV* models for the instance of the portfolio allocation problem described in Section 3; namely, when the model is constructed with 10 years of monthly returns of 11 industry portfolios. For this particular instance of the problem, the minimum and maximum possible expected portfolio returns are about 0.40, and 1.80 respectively. Let us first choose an “average” level of risk; namely, we choose $\mu_o = 1.00$ in both the MV and MV* model. In Table 2, the second and fifth columns give the resulting (optimal) MV portfolio obtained by solving (2). More important for our discussion, Figure 1 (left) shows the in-sample MV portfolio return distribution. The skewness corresponding to the MV portfolio return distribution is $-0.352$; that is, this indicates that the portfolio return distribution has substantial skewness to the left (losses). Consider an investor who wants to improve (i.e., increase) the MV portfolio’s skewness. For that purpose, we indirectly use the probability constraint in the MV* model. Specifically, we will use the probability constraint to “shorten” the left tail (i.e., making the left tail end closer to the mean value) of the MV portfolio’s return distribution. Intuitively, one would expect this shortening to result in an improvement in the skewness of the portfolio return distribution.
Table 2: MV and MV* optimal portfolios of 11 industry portfolios for $\mu_o = 1.00$, and $\alpha = -6.79$ (a position of “—” means 0.00).

<table>
<thead>
<tr>
<th>Asset</th>
<th>Portfolio (%)</th>
<th>MV</th>
<th>MV*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agric</td>
<td>13.66</td>
<td>17.54</td>
<td></td>
</tr>
<tr>
<td>Stone</td>
<td>—</td>
<td>3.61</td>
<td></td>
</tr>
<tr>
<td>Food</td>
<td>0.64</td>
<td>1.74</td>
<td></td>
</tr>
<tr>
<td>Smoke</td>
<td>4.83</td>
<td>3.00</td>
<td></td>
</tr>
<tr>
<td>Paper</td>
<td>2.97</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>Print</td>
<td>11.29</td>
<td>20.26</td>
<td></td>
</tr>
<tr>
<td>Chems</td>
<td>23.16</td>
<td>25.05</td>
<td></td>
</tr>
<tr>
<td>Ptrlm</td>
<td>12.81</td>
<td>13.13</td>
<td></td>
</tr>
<tr>
<td>Lethr</td>
<td>0.19</td>
<td>1.12</td>
<td></td>
</tr>
<tr>
<td>Utils</td>
<td>20.97</td>
<td>14.55</td>
<td></td>
</tr>
<tr>
<td>Rtail</td>
<td>9.47</td>
<td>—</td>
<td></td>
</tr>
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</table>

Notice that the left tail of the MV portfolio return distribution ends at around $-8.00$ (see Figure 1 (left)). To improve the skewness of the MV portfolio, we solve the MV* model (6) with parameters $\mu_o = 1.00$, $\alpha = \mu_o - 2.5 \times \sqrt{9.712} = -6.79$. That is, we are constraining the left tail of the portfolio return distribution to end at a value greater than or equal to $-6.79$, or 2.5 standard deviations from the mean in the original MV portfolio return distribution (notice that 9.712 is the variance of the original MV portfolio). The corresponding Quadratic Program is solved by the ILOG-CPLEX. In Table 2, the third and sixth columns give the resulting (optimal) MV* portfolio; while Figure 1 (right) shows the in-sample MV* portfolio return distribution. As required, the left tail of the MV* portfolio return distribution ends at $-6.79$, and as expected, this produces a substantial increase in the skewness of the portfolio return distribution from the original $-0.352$ to $-0.184$ (see details in Figure 1 (right)).
surprisingly, the trade-off of improving the skewness of the portfolio is an increase in the risk (variance) of the portfolio (return distribution). However, by comparing the variance of the MV portfolio: 9.712, and the MV* portfolio: 9.955, it is clear that we have managed to improve the skewness of the MV portfolio by 52% while maintaining the resulting portfolio very close to the efficient frontier.

Notice that the parameters of the MV* model are simple and intuitive: $\mu_o$ is the typical MV parameter of risk preference; while $\alpha$ defines the intuitively simple probability constraint in (5). Moreover, by comparing the optimal MV and MV* portfolios in Table 2, it is clear that the amount of rebalancing needed to go from the MV to the MV* portfolio is definitely not high. Moreover, notice that unlike related models proposed to improve the skewness of MV portfolios (see, e.g., Boyle and Ding (2005); and Konno et al. (1993)), no iterative process, or ad-hoc approximations and parameters are used to obtain the MV* portfolio. In fact, the MV* optimal portfolio can be quickly obtained by directly solving the straight-forward formulation of the MV* model (6) with a commercial optimization or financial solver such as ILOG-CPLEX, Excel Solver, Matlab, or BARRA to name a few. Therefore, the MV* model is a simple tool that can be used to improve the skewness of portfolios on the mean-variance efficient frontier.

Figure 1: In-sample return distributions of MV (left) and MV* (right) optimal portfolios of 11 industry portfolios for $\mu_o = 1.00$, and $\alpha = -6.79$. 

![Figure 1](image-url)
To provide further evidence of how the MV* model can be used to improve the skewness of the MV portfolios while maintaining the resulting portfolios close to the efficient frontier, we now replicate the experiment described above through out the MV efficient frontier. For this purpose, we first calculate the MV efficient frontier (see Figure 2 (right)) by solving the MV model for twenty (20) different levels of minimum expected return \( \mu_0 \). Then we solve the MV* model for the same levels of minimum expected return, with \( \alpha = \mu_0 - 2.5 \times \sigma^{MV}(\mu_0) \), where \( \sigma^{MV}(\mu_0) \) indicates the square root of the variance of the MV portfolio with required expected return \( \mu_0 \). Thus, we intend to improve the skewness of the portfolios through the MV efficient frontier by shortening the left tail of the portfolio return distribution to 2.5 standard deviations from the mean. The results are presented in Figure 2. Notice that as in our first experiment with \( \mu_0 = 1.00 \), we obtain the desired improvement on the skewness of the MV portfolios by solving the MV* model (see Figure 2 (right)). Moreover, the resulting MV* portfolios are close to the MV efficient frontier (see Figure 2 (left)). Therefore, all the characteristics of the MV* model discussed in this section hold through out the MV efficient frontier.

Figure 2: Efficient frontier (left) and skewness (right) of MV and MV* optimal portfolios, with \( \alpha = \mu_0 - 2.5\sigma^{MV}(\mu_0) \).
4.1 Aggressive improvement of skewness

Thus far in this section, we have concentrated in improving the skewness of MV portfolios while maintaining the resulting portfolio close to the mean-variance efficient frontier. Consider for a moment an investor that is willing to get further away from the mean-variance efficient frontier with the objective of getting a portfolio with much higher skewness, compared to the improvements made in our previous experiments of this section. For that purpose, let us revisit the experiment at the beginning of the section with $\mu_o = 1.00$. This time however; we make a more aggressive shortening of the left tail of the MV portfolio return distribution, by setting $\alpha = -6.00$ (instead of $\alpha = -6.79$ used at the beginning of the section) in the MV* model. The corresponding results are shown in Table 3, and Figure 3. From Figure 3 (right), it is clear that the aggressiveness has paid off, as we have increased the skewness from -0.352 in the MV portfolio, to 0.209 in the MV* portfolio. Not only is the skewness of the MV* portfolio positive, but the range of the distribution is clearly superior to that of the original MV portfolio (compare Figure 3 (left), and Figure 3 (right)). However, to obtain this great improvement in skewness it is necessary to make a significant tradeoff in variance; that is, the variance of the resulting MV* portfolio is 11.59; which is fairly higher than the MV portfolio variance of 9.71. This means that the MV* portfolio is not as close to the efficient frontier as in our previous experiments, and the investor would have to decide if the trade-off between skewness and risk (variance) in this case is worthy of considering the aggressive MV* portfolio. Also, it is not surprising that obtaining this great improvement in skewness requires substantial rebalancing, in case the investor wants to move from the MV portfolio to the aggressive MV* portfolio (see Table 3). The importance of this experiment is that it further shows that the MV* model is very flexible, and can be used in different ways depending on the investor’s needs.

5 Final Remarks
Table 3: MV and MV* optimal portfolios of 11 industry portfolios for $\mu_o = 1.00$, and $\alpha = -6.00$ (a position of “—” means 0.00).

<table>
<thead>
<tr>
<th>Asset</th>
<th>Portfolio (%)</th>
<th>MV</th>
<th>MV*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agric</td>
<td>13.66</td>
<td>36.27</td>
<td></td>
</tr>
<tr>
<td>Stone</td>
<td>—</td>
<td>1.18</td>
<td></td>
</tr>
<tr>
<td>Food</td>
<td>0.64</td>
<td>—</td>
<td></td>
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<tr>
<td>Smoke</td>
<td>4.83</td>
<td>6.34</td>
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</tr>
<tr>
<td>Rtail</td>
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</table>

Figure 3: In-sample return distributions of MV (left) and MV* (right) optimal portfolios of 11 industry portfolios for $\mu_o = 1.00$, and $\alpha = -6.00$. 
We have presented a MV$^*$ model that, similar to the work of Boyle and Ding (2005); and Konno et al. (1993), improves the skewness of portfolios in the mean-variance efficient frontier. Unlike the work of Boyle and Ding (2005); and Konno et al. (1993), the MV$^*$ model is very intuitive, and non-iterative. Also, the MV$^*$ model is simple to implement and carry out. These characteristics should make our methodology very appealing to both practitioners and researchers in risk and portfolio management, and specially appealing to actuaries who face asset/liability allocation problems which very sparse historical data is available, and that due to regulations and the importance of catastrophic events in actuarial science, are interested in controlling the left tail (high losses) of the portfolio return distribution.

To finish, let us make some final remarks:

- Although in our numerical experiments we used (3) and (4) to obtain $\mu$, and $\Sigma$, our approach is independent of this choice. For example, to compute these estimates one could use other methods; such as the one proposed by Black and Litterman (2001); and Meucci (2006), to take into account the investors views about the assets; or the one proposed by Cavadini et al. (2001); and DeMiguel and Nogales (2006), to obtain robust estimates for the variance-covariance of the asset returns. Also, resampling methodologies as the one proposed by Michaud (1998) can be used within the framework of the MV$^*$ methodology proposed here.

- Notice that linear diversification constraints can be added to the model to take into account other features not considered here such as: transaction costs, benchmark, and liquidity constraints; without changing the properties of the MV$^*$ model.

- Notice that as long as $\mu_o$ is less than $\mu^*$: the expected return of the asset with maximum expected return, the MV model is always feasible. Clearly, this might not be the case for the MV$^*$ model; that is, even with $\mu_o \leq \mu^*$ certain choices $\alpha, \beta$, might lead to the MV$^*$ model being infeasible. For example, when $\mu_o$ is close to $\mu^*$ the portfolios will have a low number of positions, and there will be...
be very little freedom on the VaR-like constraints that can be enforced on the portfolio return distribution. For this reason, in our experiments we computed the efficient frontier up to values of $\mu_0 \leq 0.80\mu_\ast$.

- We want to mention that our experiments show that the very popular (both in practice an academia) CVaR model of Rockafellar and Uryasev (2000) produces portfolios with higher skewness than the corresponding mean-variance portfolio (for the same portfolio expected return $\mu_0$). However, CVaR portfolios are typically far from the mean-variance efficient frontier. Following the work of Krokhmal et al. (2002), we are currently looking at using CVaR as an alternative way to improve the skewness of portfolios in the mean-variance efficient frontier.

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**References**


